Operator Theory
Advances and Applications

Birgit Jacob
Hans J. Zwart

Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces

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Preface

The aim of this book is to give a self-contained introduction to the theory of infinite-dimensional systems theory and its applications to port-Hamiltonian systems.

The field of infinite-dimensional systems theory has become a well-established field within mathematics and systems theory. There are basically two approaches to infinite-dimensional linear systems theory: an abstract functional analytical approach and a PDE approach. There are excellent books dealing with infinite-dimensional linear systems theory, such as (in alphabetical order) Bensoussan, Da Prato, Delfour and Mitter [6], Curtain and Pritchard [9], Curtain and Zwart [10], Fattorini [17], Luo, Guo and Morgul [40], Lasiecka and Triggiani [34, 35], Lions [37], Lions and Magenes [38], Staffans [51], and Tucsnak and Weiss [54].

Many physical systems can be formulated using a Hamiltonian framework. This class contains ordinary as well as partial differential equations. Each system in this class has a Hamiltonian, generally given by the energy function. In the study of Hamiltonian systems it is usually assumed that the system does not interact with its environment. However, for the purpose of control and for the interconnection of two or more Hamiltonian systems it is essential to take this interaction with the environment into account. This led to the class of port-Hamiltonian systems, see [56, 57]. The Hamiltonian/energy has been used to control a port-Hamiltonian system, see e.g. [4, 7, 21, 43]. For port-Hamiltonian systems described by ordinary differential equations this approach is very successful, see the references mentioned above. Port-Hamiltonian systems described by partial differential equations is a subject of current research, see e.g. [14, 28, 33, 41].

In this book, we combine the abstract functional analytical approach with the more physical approach based on Hamiltonians. For a class of linear infinite-dimensional port-Hamiltonian systems we derive easily verifiable conditions for well-posedness and stability.

The material of this book has been developed over a series of years. Javier Villegas [58] studied in his PhD-thesis a port-Hamiltonian approach to distributed parameter systems. We are grateful to Javier Villegas that we could include his results into the book. The first setup of the book was written for a graduate course on control of distributed parameter systems for the Dutch Institute of Systems and Control (DISC) in the spring of 2009 which was attended by 25 PhD students. This material was adapted for the CIMPA-UNESCO-Marrakech School on Control and Analysis for PDE in May 2009. In 2010-2011 we were the virtual lecturers of the 14th Internet Seminar on Infinite-dimensional Linear Systems Theory. More than 300 participants attended this virtual course and a wikipage was used to discuss the material and to post typos and comments. For this course we decided to add extra chapters on finite-dimensional systems theory, and to make the material in the later chapters more accessible.

We are indebted to the help from many colleagues and friends. We are grateful to the participants of the DISC-course, the CIMPA-UNESCO-Marrakesch School and the 14th Internet Seminar for their useful comments and questions. Large parts of the manuscript have been read by our colleagues Mikael Kurula (Twente) and Christian Wyss (Wuppertal), who made many useful comments for improvements.

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Birgit Jacob and Hans Zwart,
November 2011
Wuppertal and Twente

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Chapter 1

Introduction

In this chapter we provide an introduction to the field of mathematical systems theory. Besides examples we discuss the notion of feedback and we answer the question why feedback is useful. However, before we start with the examples we discuss the following picture, which can be seen as the essence of systems theory. In

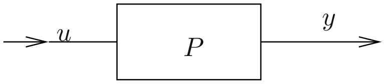

Figure 1.1: A system

systems theory, we consider models which are in contact with their environment. In the above picture, P denotes the to-be-studied-model which interacts with its environment via u and y. u and y are time signals, i.e., functions of the time t. The function u denotes the signal which influences P and y is the signal which we observe from P. u is called the *input* or *control* and y is called the *output*. The character P is chosen, since it is short for plant, think of chemical or power plant. To obtain a better understanding for the general setting as depicted in Figure 1.1, we discuss several examples in which we indicate the input and output.

Regarding these examples, we should mention that we use different notations for derivatives. In the best tradition of mathematics, we use $\dot{f}$, $\frac{df}{dt}$, and $f^{(1)}$ to denote the first derivative of the function f. Similarly for higher derivatives.

1.1 Examples

Example 1.1.1. Newton's second law states that the force applied to a body produces a proportional acceleration; that is

$$F(t) = m\ddot{q}(t). \tag{1.1}$$

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*,
Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_1,

Here $q(t)$ denotes the position at time t of the particle with mass m, and $F(t)$ is the force applied to it. Regarding the external force $F(t)$ as our input $u(t)$ and choosing the position $q(t)$ as our output $y(t)$, we obtain the differential equation

$$\ddot{y}(t) = \frac{1}{m}u(t), \qquad t \geq 0. \tag{1.2}$$

Thus the differential equation describes the behaviour "inside the box P", see Figure 1.1. Further, we can influence the system via the external force u and we observe the position y. In this simple example we clearly see that u is not the only quantity that determines y. The output also depends on the initial position $q(0)$ and the initial velocity $\dot{q}(0)$. They are normally not at our disposal to choose freely, and so they are also "inside the box".

Example 1.1.2. Consider the electrical network given by Figure 1.2. Here V denotes the voltage source, L_1, L_2 denote the inductance of the inductors, and C denotes the capacitance of the capacitor.

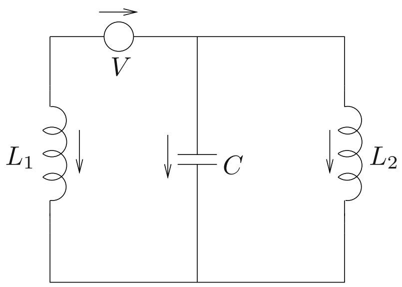

Figure 1.2: Electrical network

For the components in an electrical circuit, the following basic laws hold: the current I_L and voltage V_L across an inductor with inductance L is related via

$$V_L(t) = L\frac{dI_L}{dt}(t), \tag{1.3}$$

whereas the current I_C and voltage V_C of the capacitor with capacitance C are related via

$$I_C(t) = C\frac{dV_C}{dt}(t). \tag{1.4}$$

The conservation of charge and energy in electrical circuits are described by Kirchhoff's circuit laws. Kirchhoff's first law states that at any node in an electrical circuit, the sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Moreover, Kirchhoff's second law says that the directed sum of the electrical potential differences around any closed circuit must be zero.

Applying these laws to our example, we obtain the following differential equations:

$$L_1 \frac{dI_{L_1}}{dt}(t) = V_{L_1}(t) = V_C(t) + V(t), \tag{1.5}$$

$$L_2 \frac{dI_{L_2}}{dt}(t) = V_{L_2}(t) = V_C(t), \qquad \text{and} \tag{1.6}$$

$$C \frac{dV_C}{dt}(t) = I_C(t) = -I_{L_1}(t) - I_{L_2}(t). \tag{1.7}$$

We assume that we can only measure the current I_{L_1}, i.e. we define

$$y(t) = I_{L_1}(t).$$

Using (1.5), we obtain $L_1 y^{(1)}(t) = V_C(t) + V(t)$. Further, (1.7) then implies

$$L_1 C y^{(2)}(t) = -y(t) - I_{L_2}(t) + CV^{(1)}(t). \tag{1.8}$$

Differentiating this equation once more and using (1.6), we find

$$\begin{aligned} L_1 C y^{(3)}(t) &= -y^{(1)}(t) - \frac{dI_{L_2}}{dt}(t) + CV^{(2)}(t) = -y^{(1)}(t) - \frac{1}{L_2} V_C(t) + CV^{(2)}(t) \\ &= -y^{(1)}(t) - \frac{1}{L_2}\left(L_1 y^{(1)}(t) - V(t)\right) + CV^{(2)}(t), \end{aligned} \tag{1.9}$$

where we have used (1.5) as well. We regard the voltage supplied by the voltage source as the input u. Thus we obtain the following ordinary differential equation describing our system:

$$L_1 C y^{(3)}(t) + \left(1 + \frac{L_1}{L_2}\right) y^{(1)}(t) = \frac{1}{L_2} u(t) + C u^{(2)}(t). \tag{1.10}$$

Example 1.1.3. Suppose we have a mass m which can move along a line, as depicted in Figure 1.3. The mass is connected to a spring with spring constant k, which in turn is connected to a wall. Furthermore, the mass is connected to a damper whose (friction) force is proportional to the velocity of the mass by the constant r. The third force which is working on the mass is given by the external force $F(t)$.

Let $q(t)$ be the distance of the mass to the equilibrium point. Then by Newton's law we have that

$$m\ddot{q}(t) = \text{total sum of the forces.}$$

As the force of the spring equals $kq(t)$ and the force by the damper equals $r\dot{q}(t)$, we find

$$m\ddot{q}(t) + r\dot{q}(t) + kq(t) = F(t). \tag{1.11}$$

We regard the external force $F(t)$ as our input u and we choose the position as our output y. This choice leads to the differential equation

$$m\ddot{y}(t) + r\dot{y}(t) + ky(t) = u(t). \tag{1.12}$$

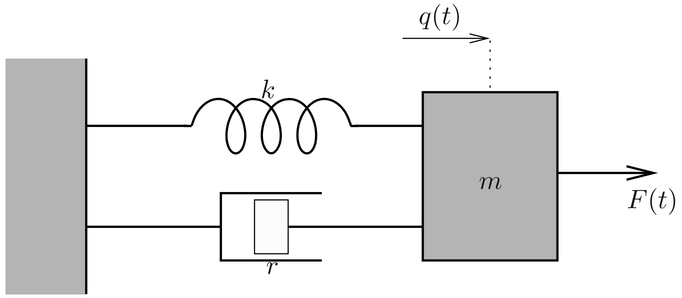

Figure 1.3: Mass-spring-system

Until now we have only seen examples which can be modelled by ordinary differential equations. The following examples are modelled via a partial differential equation.

Example 1.1.4. We consider the *vibrating string* as depicted in Figure 1.4. The

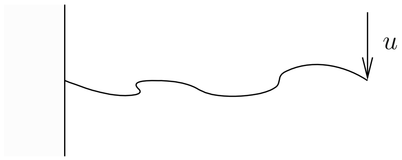

Figure 1.4: The vibrating string

string is fixed at the left-hand side and may move freely at the right-hand side. We allow that a force u may be applied at that side. The model of the (undamped) vibrating string is given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right) \tag{1.13}$$

where $\zeta \in [a, b]$ is the spatial variable, $w(\zeta, t)$ is the vertical displacement of the string at position ζ and time t, T is the Young's modulus of the string, and ρ is the mass density, which may vary along the string. This model is a simplified version of other systems where vibrations occur, as in the case of large structures, and it is also used in acoustics. The partial differential equation (1.13) is also known as the *wave equation*.

If the mass density and the Young's modulus are constant, then we get the partial differential equation

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = c^2 \frac{\partial^2 w}{\partial \zeta^2}(\zeta, t), \tag{1.14}$$

where $c^2 = T/\rho$. This is the most familiar form of the wave equation.

In contrast to ordinary differential equations, we need boundary conditions for our partial differential equation (1.13) or (1.14). At the left-hand side, we put the position to zero, i.e.,

$$w(a,t) = 0 \tag{1.15}$$

and at the right-hand side, we have the balance of the forces, which gives

$$T(b)\frac{\partial w}{\partial \zeta}(b,t) = u(t). \tag{1.16}$$

There are different options for the output. One option is to measure the velocity at the right-hand side, i.e.,

$$y(t) = \frac{\partial w}{\partial t}(b,t). \tag{1.17}$$

Another option could be to measure the velocity at a point between a and b, or to measure the position of the wave, i.e., $y(t) = w(\cdot,t)$. Hence at every time instant, y is a function of the spatial coordinate.

We end this section with another well-known partial differential equation.

Example 1.1.5. The model of *heat conduction* consists of only one conservation law, that is, the *conservation of energy*. It is given as

$$\frac{\partial e}{\partial t} = -\frac{\partial}{\partial \zeta} J_Q, \tag{1.18}$$

where $e(\zeta,t)$ is the energy density and $J_Q(\zeta,t)$ is the heat flux. This conservation law is completed by two closure equations. The first one expresses the calorimetric properties of the material:

$$\frac{\partial e}{\partial T} = c_V(T), \tag{1.19}$$

where $T(\zeta,t)$ is the temperature distribution and c_V is the heat capacity. The second closure equation defines the heat conduction property of the material (Fourier's conduction law):

$$J_Q = -\lambda(T,\zeta)\frac{\partial T}{\partial \zeta}, \tag{1.20}$$

where $\lambda(T,\zeta)$ denotes the heat conduction coefficient. Assuming that the variations of the temperature are not too large, we may assume that the heat capacity and the heat conduction coefficient are independent of the temperature. Thus we obtain the partial differential equation

$$\frac{\partial T}{\partial t}(\zeta,t) = \frac{1}{c_V}\frac{\partial}{\partial \zeta}\left(\lambda(\zeta)\frac{\partial T}{\partial \zeta}(\zeta,t)\right), \qquad \zeta \in (a,b),\ t \geq 0. \tag{1.21}$$

As for the vibrating string, the constant coefficient case is better known. This is

$$\frac{\partial T}{\partial t}(\zeta,t) = \alpha \frac{\partial^2 T}{\partial \zeta^2}(\zeta,t), \qquad \zeta \in (a,b),\ t \geq 0 \tag{1.22}$$

with $\alpha = \lambda / c_V$.

Again, we need boundary conditions for the partial differential equations (1.21) and (1.22). If the heat conduction takes places in a perfectly insulated surrounding, then no heat can flow in or out of the system, and we have as boundary conditions

$$\lambda(a)\frac{\partial T}{\partial \zeta}(a,t) = 0, \text{ and } \lambda(b)\frac{\partial T}{\partial \zeta}(b,t) = 0. \tag{1.23}$$

It can also be that the temperature at the boundary is prescribed. For instance, if the ends are lying in a bath with melting ice, then we obtain the boundary conditions

$$T(a,t) = 0, \text{ and } T(b,t) = 0. \tag{1.24}$$

As measurement we can take the temperature at a point $y(t) = T(\zeta_0, t), \zeta_0 \in (a,b)$. Another choice could be the average temperature in an interval (ζ_0, ζ_1). In the latter case we find

$$y(t) = \frac{1}{\zeta_1 - \zeta_0} \int_{\zeta_0}^{\zeta_1} T(\zeta,t)\, d\zeta.$$

As input we could control the temperature at one end of the spatial interval, e.g. $T(b,t) = u(t)$, or we could heat it in the interval (ζ_0, ζ_1). The latter choice leads to the partial differential equation

$$\frac{\partial T}{\partial t}(\zeta,t) = \frac{1}{c_V}\frac{\partial}{\partial \zeta}\left(\lambda(\zeta)\frac{\partial T}{\partial \zeta}(\zeta,t)\right) + u(\zeta,t),$$

where we define $u(\zeta,t) = 0$ for $\zeta \notin (\zeta_0, \zeta_1)$.

1.2 How to control a system?

In the previous section we have seen that there are many models in which we can distinguish an input and an output. Via the input we have the possibility to influence the system. In particular, we aim to choose the input u such that y or all variables in the box behave as we desire. Note that the phrase "to behave as we desire" means that we have to make choices. These choices will be based on the type of plant we are dealing with. If P represents a passenger airplane, and y the height, we do not want y to go from 10 kilometers to ground level in one second. However, we would like that this happens in half an hour. On the other hand, if the plant represents a stepper, then very fast action is essential. A stepper is a device used in the manufacture of integrated circuits (ICs); it got its name from the fact that it moves or "steps" the silicon wafer from one shot location to another. This has to be done very quickly, and with a precision on nano-meters.

So generally, the control task is to find an input u such that y has a desired behaviour and we shall rarely do it by explicitly calculating the function u. More often we design u on the basis of y. Hence instead of open loop systems described by Figure 1.1, here given once more as Figure 1.5, we work with closed loop systems given by Figure 1.6. In order to "read" the latter picture, it is sufficient to know

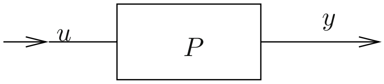

Figure 1.5: Our system

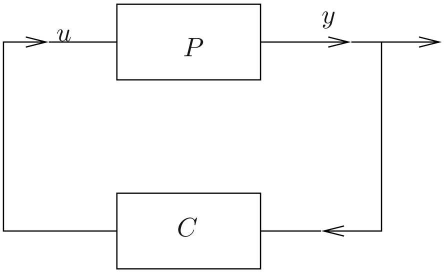

Figure 1.6: Feedback system

some simple rules. As before, by a rectangular block we denote a model relating the incoming signal to the outgoing signal. This could for instance be an ordinary or partial differential equation. Mathematically speaking one may see P and C as operators mapping the signal u to the signal y and vice versa. At a node we assume that the incoming signals are the same as the outgoing signals. The arrows indicate the directions of the signals.

As before we denote by P the system that we desire to control, and by C we denote our (designed) controller.

In this section we show that closed loop systems have in general better properties than open loop systems. We explain the advantages by means of an example. We consider the simple control problem of steering the position of the mass m to zero, see Example 1.1.1. Hence, the control problem is to design the force F such that, for every initial position and every initial velocity of the mass, the position of the mass is going to zero for time going to infinity. To simplify the problem even more we assume that the mass m equals 1, and we assume that we measure the velocity and the position, i.e.,

$$\ddot{q}(t) = u(t) \quad \text{and} \quad y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \tag{1.25}$$

In Exercise 1.2 the case $y(t) = q(t)$ is discussed.

In order to design the controller C for the plant described by (1.25) we proceed as follows. First we have to specify what we mean by "to behave as we desire", that is, we have to decide how and how fast the position should go to zero. It is quite common to choose an "ideal" differential equation such that the solutions have the desired behaviour. We consider the solutions of the differential equation

$$f^{(2)}(t) + 2f^{(1)}(t) + f(t) = 0. \tag{1.26}$$

The general solution of (1.26) is given by $f(t) = \alpha e^{-t} + \beta te^{-t}$. This is to our satisfaction, and so we try to design a control u such that the position q is equal to a solution of the equation (1.26). If we choose

$$u(t) = \begin{bmatrix} -1 & -2 \end{bmatrix} y(t) = -2q^{(1)}(t) - q(t), \tag{1.27}$$

then the position of the mass indeed behaves in the same way as the solutions of (1.26). Note that for the design of this feedback law no knowledge of the initial position nor the initial velocity is needed. If we want to obtain the same behavior of the solution by an open loop control, i.e., if we want to construct the input as an explicit function of time, we need to know the initial position and the initial velocity. Suppose they are given as $q(0) = -2$ and $q^{(1)}(0) = 5$, respectively. Calculating the position as the solution of

$$q^{(2)}(t) + 2q^{(1)}(t) + q(t) = 0, \quad t \geq 0, \qquad q(0) = -2, \quad q^{(1)}(0) = 5,$$

we obtain

$$q(t) = -2e^{-t} + 3te^{-t}. \tag{1.28}$$

Thus by (1.27) we find

$$u(t) = -8e^{-t} + 3te^{-t}. \tag{1.29}$$

Applying this input to the system

$$q^{(2)}(t) = u(t) \tag{1.30}$$

would give the same behavior as applying (1.27).

We simulate the open loop system, i.e., (1.30) with $u(t)$ given by (1.29) and the closed loop system, i.e, (1.30) with $u(t)$ given by (1.27). The result is shown in Figure 1.7. We obtain that the simulation of the open loop system is worse than the one of the closed loop system. This could be blamed on a bad numerical solver, but even mathematically, we can show that the closed loop system behaves in a superior manner to the open loop system. We have assumed that we know the initial data exactly, but this will never be the case. So suppose that we have (small) errors in both initial conditions, but we are unaware of the precise error. Thus we apply the input (1.29) to the system

$$q^{(2)}(t) = u(t), \quad t \geq 0, \qquad q(0) = q_0, \quad q^{(1)}(0) = q_1,$$

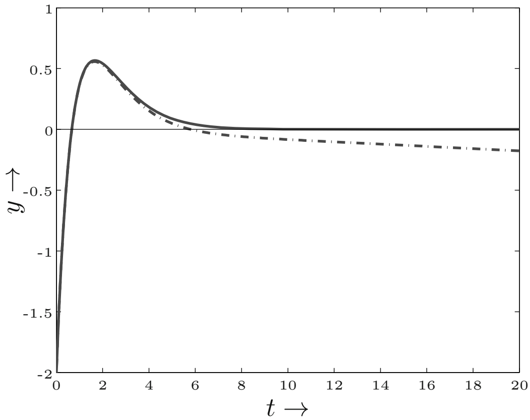

Figure 1.7: The solution of the open loop system (1.30) and (1.29) (dashed line) and the solution of the closed loop system (1.30) and (1.27) (solid line)

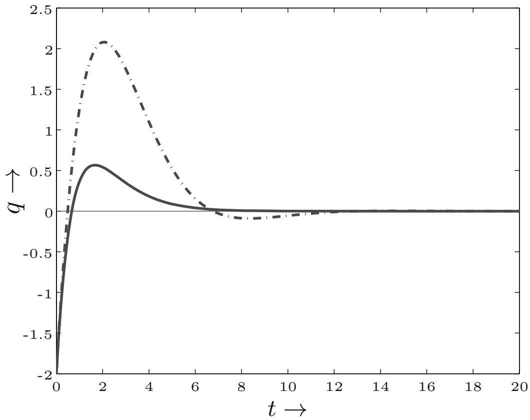

Figure 1.8: The solution of the system $2q^{(2)}(t) = u(t)$ with feedback (1.27) (dashed line) and the solution of the system (1.30) with the same feedback (1.27) (solid line)

where q_0 and q_1 are not exactly known initial conditions. The solution of this ordinary differential equation is given by

$$q(t) = -2e^{-t} + 3te^{-t} + (q_0 + 2) + (q_1 - 5)t. \tag{1.31}$$

Hence any small error in $q(0)$ will remain, and any error in $q^{(1)}(0)$ will even increase. This effect also occurred in the simulation. No matter how good the quality of the simulation is, there will always be small errors, which results in the misfit of the solution.

Apart from the initial conditions which could contain a (small) error, there is always the question of the exact value of the physical parameters. To illustrate this, assume that we have measured the mass with a large error, so assume that the real value of the mass is 2. By means of a simulation, we show that the feedback law (1.27) still works. Note, the function q can also be calculated analytically. The position is still converging to zero, and so the design criterion is still satisfied. This shows the power of feedback.

1.3 Exercises

1.1. Consider the electrical network as given below.

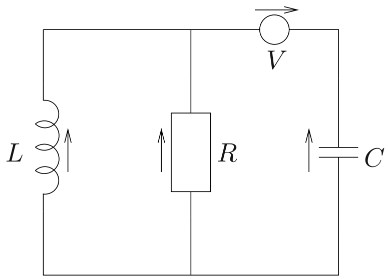

Figure 1.9: RCL network

As input we choose the voltage supplied by the voltage source, and as output we measure the voltage over the resistor. Note that the current I_R and the voltage V_R across a resistor with resistance R is related via $V_R = RI_R$. Determine the ordinary differential equation modeling the relation between u and y.

1.2. As in Section 1.2 we study the system

$$\ddot{q}(t) = u(t). \tag{1.32}$$

In this and the next exercise we investigate the properties of the feedback a little further. First we show that the feedback as designed in Section 1.2 still works well if we add some perturbation to the output. In the second part we study whether the same holds for the open loop system.

(a) As output we choose

$$y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}.$$

However, we assume that we cannot observe the output exactly due to some noise. We model the noise as a small signal which changes quickly, and so we take $\begin{bmatrix} \varepsilon\sin(\omega t) \\ \varepsilon\sin(\omega t) \end{bmatrix}$, with ε small, and ω large. For simplicity we have taken both components of the noise to be equal, but this is non-essential.

So the output that we have at our disposal for controlling the system is

$$\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} + \begin{bmatrix} \varepsilon\sin(\omega t) \\ \varepsilon\sin(\omega t) \end{bmatrix}.$$

We apply the same feedback law as given in (1.27), that is,

$$u(t) = -2(q^{(1)}(t) + \varepsilon\sin(\omega t)) - (q(t) + \varepsilon\sin(\omega t)).$$

Determine the solution of the closed loop system, and conclude that the perturbation on the output y remains bounded. More precisely, the perturbation on the output y is bounded by $C\frac{\varepsilon}{\omega}$, where $C > 0$ is independent of $\varepsilon > 0$ and $\omega \geq 1$.

(b) In the previous part we have seen that adding noise to the observation or control hardly effects the desired behavior when we work with feedback systems. We now investigate if the same holds for the open loop system. We consider the system (1.32) with initial conditions $q(0) = -2, \dot{q}(0) = 5$, as input we take

$$u(t) = -8e^{-t} + 3te^{-t} - 3\varepsilon\sin(\omega t)$$

and as output we choose

$$y(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}.$$

Calculate the output y and conclude that the perturbation on the output y is unbounded.

1.3. We consider again the system described by equation (1.32), but now we assume that we measure the position only, i.e., $y(t) = q(t)$.

(a) As feedback we choose $u(t) = ky(t)$, $k \in \mathbb{R}$, where $y(t) = q(t)$.
Show that for every $k \in \mathbb{R}$ there exists an initial condition such that the solutions of the differential equation (1.32) with this feedback do not converge to zero.

(b) As in the previous part we choose the output $y(t) = q(t)$, but we model the controller C again by a differential equation, that is, the input u is an output of another system. More precisely, we choose

$$\dot{z}(t) = -3z(t) + 8y(t), \qquad u(t) = z(t) - 3y(t). \tag{1.33}$$

Show that if y, z, u satisfies

$$\begin{aligned} \dot{z}(t) &= -3z(t) + 8y(t), \\ u(t) &= z(t) - 3y(t), \\ \ddot{y}(t) &= u(t), \end{aligned}$$

then y satisfies the equation

$$y^{(3)}(t) = z^{(1)}(t) - 3y^{(1)}(t) = -3\left(y^{(2)}(t) + 3y(t)\right) + 8y(t) - 3y^{(1)}(t).$$

Conclude that y converges exponentially to zero. More precisely, we have $|y(t)| \leq C(t^2+1)e^{-t}$, for some constant $C > 0$.

1.4 Notes and references

The examples and results presented in this chapter can be found in many books on systems theory. We refer to [50] and [32] for the examples described by ordinary differential equations, and to [10] for the examples described by partial differential equations. [50] also provides a discussion on feedback systems.

Chapter 2

State Space Representation

In the previous chapter we introduced models with an input and an output. These models were described by an ordinary or partial differential equation. However, there are other possibilities to model systems with inputs and outputs. In this chapter we introduce the state space representation on a finite-dimensional state space. Later we will encounter these representations on an infinite-dimensional state space. State space representations enable us to study systems with inputs and outputs in a uniform framework. In this chapter, we show that every model described by an ordinary differential equation possesses a state space representation on a finite-dimensional state space, and that it is just a different way of writing down the system. However, this different representation turns out to be very important as we will see in the following chapters. In particular, it enables us to develop general control strategies.

2.1 State space models

In this section, we show that every linear ordinary differential equation with constant coefficients can be written in the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \tag{2.1}$$

where x is a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and A, B, C, and D are matrices of appropriate sizes. This is known as the *state space representation*, *state space system* or *state space model*. The vector x is called the *state*. State space representations with a finite-dimensional state space are also called *finite-dimensional systems*. The first equation of (2.1) is named the *state differential equation*.

Note that most of the calculations in this section are formally, that is, we do not need the notion of (classical) solution of linear ordinary differential equations with constant coefficients nor of a state space representation. Further, we have no specific assumption on the input u. One may for simplicity just assume that

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_2,

u is sufficiently smooth. The aim of this section is to show that under some mild conditions state space representations and linear ordinary differential equations with constant coefficients are equivalent. Note that, under mild assumptions, it can be shown that both representations possess the same solutions. In the next section, we discuss the notion of classical and mild solutions of the state space representation (2.1) in more details.

First of all we show that the Examples 1.1.1 and 1.1.2 have a state space representation.

Example 2.1.1. Consider Newton's law of Example 1.1.1, with input the force and output the position, see equation (1.2),

$$u(t) = m\ddot{y}(t).$$

We choose the state $x(t) \in \mathbb{R}^2$ as

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}.$$

For this choice we see that

$$\dot{x}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t). \tag{2.2}$$

Thus Newton's law can be written in the standard state space formulation with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } D = 0.$$

For Newton's law we had to introduce a new state variable in order to obtain the state space representation. However, some models appear naturally in the state space representation. This holds for example for the electrical circuit of Example 1.1.2.

Example 2.1.2. Consider the electrical circuit of Example 1.1.2. The differential equation relating the voltage V provided by the voltage source, and the current through the first inductor I_{L_1} is given by the differential equation

$$L_1 C y^{(3)}(t) + \left(1 + \frac{L_1}{L_2}\right) y^{(1)}(t) = \frac{1}{L_2} u(t) + C u^{(2)}(t).$$

However, to find the state space representation we use the equations (1.5)–(1.7) directly, that is, we consider the equations

$$L_1 \frac{dI_{L_1}}{dt}(t) = V_{L_1}(t) = V_C(t) + V(t), \tag{2.3}$$

$$L_2 \frac{dI_{L_2}}{dt}(t) = V_{L_2}(t) = V_C(t), \qquad \text{and} \tag{2.4}$$

$$C \frac{dV_C}{dt}(t) = I_C(t) = -I_{L_1}(t) - I_{L_2}(t). \tag{2.5}$$

As state vector we choose

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} I_{L_1}(t) \\ I_{L_2}(t) \\ V_C(t) \end{bmatrix}.$$

Using the equations (2.3)–(2.5) we find

$$\dot{x}(t) = \begin{bmatrix} \frac{1}{L_1}x_3(t) + \frac{1}{L_1}V(t) \\ \frac{1}{L_2}x_3(t) \\ -\frac{1}{C}x_1(t) - \frac{1}{C}x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} V(t). \tag{2.6}$$

If we use the (standard) notation u for the input V and y for the output/measurement I_{L_1}, then the system of Example 1.1.2 can be written in the state space representation with

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \text{ and } D = 0.$$

Our construction of the state space representation may seem rather ad-hoc. However, next we show that every linear ordinary differential equation with constant coefficients has a state space representation. In order to explain the main ideas we first consider an example.

Example 2.1.3. Consider the following differential equation.

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 7\dot{u}(t) + 8u(t). \tag{2.7}$$

The standard way of deriving a state space representation from a differential equation is to transform it into an integral equation.

In order to this we first move every term, except the highest derivative of y, to the right-hand side of the equation. Thus we obtain,

$$\ddot{y}(t) = 7\dot{u}(t) + 8u(t) - 5\dot{y}(t) - 6y(t).$$

Now we integrate this equation as often as needed to remove all the derivatives of y, here we have to integrate twice

$$\begin{aligned} y(t) &= \int_0^t \int_0^s \big(7\dot{u}(\tau) + 8u(\tau) - 5\dot{y}(\tau) - 6y(\tau)\big)\, d\tau\, ds + y(0) + \dot{y}(0)t \\ &= \int_0^t \left(7u(s) - 5y(s) - 7u(0) + 5y(0) + \int_0^s 8u(\tau) - 6y(\tau)\, d\tau\right) ds + y(0) + \dot{y}(0)t \\ &= y(0) + \int_0^t \left(7u(s) - 5y(s) - 7u(0) + 5y(0) + \dot{y}(0) + \int_0^s 8u(\tau) - 6y(\tau)\, d\tau\right) ds. \end{aligned} \tag{2.8}$$

Note that we only have to evaluate two integrals: the integral of $-6y(t) + 8u(t)$, and the integral of the first integral plus $7u(t) - 5y(t) - 7u(0) + 5y(0) + \dot{y}(0)$. Up to a constant, we choose the result of these integrals as our state variables, i.e., $x_k(t)$, $k = 1, 2$,

$$y(t) = \underbrace{\int_0^t \Bigg(7u(s) - 5y(s) \underbrace{-7u(0) + 5y(0) + \dot{y}(0) + \int_0^s 8u(\tau) - 6y(\tau)\, d\tau}_{x_2(s)} \Bigg) ds}_{x_1(t)} + y(0)\,.$$

With this choice of variables we obtain

$$\begin{cases} y(t) = x_1(t) \\ \dot{x}_1(t) = 7u(t) - 5y(t) + x_2(t) \\ \dot{x}_2(t) = 8u(t) - 6y(t) \end{cases},$$

or equivalently

$$\dot{x}(t) = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 7 \\ 8 \end{bmatrix} u(t), \tag{2.9}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \tag{2.10}$$

This is a state space representation of the differential equation (2.7).

For a general differential equation with constant coefficients we can adopt the procedure used in the previous example. In order not to drown in the notation and formulas we omit the variable t. Consider the differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y^{(1)} + p_0 y = q_m u^{(m)} + q_{m-1}u^{(m-1)} + \cdots + q_1 u^{(1)} + q_0 u. \tag{2.11}$$

We assume that $n \geq m$. Note that without loss of generality, we may assume that $m = n$, otherwise we may choose $q_{m+1} = \cdots = q_n = 0$. Moving every y-term, except the highest derivative of y, to the right-hand side gives

$$y^{(n)} = q_n u^{(n)} + \left(q_{n-1}u^{(n-1)} - p_{n-1}y^{(n-1)} \right) + \cdots + \left(q_1 u^{(1)} - p_1 y^{(1)} \right) + (q_0 u - p_0 y)\,.$$

In order to explain the main ideas we ignore the initial conditions in the following. Integrating this equation n times and reordering the terms, we obtain

$$y = q_n u + \int \left(q_{n-1}u - p_{n-1}y + \cdots + \int \left(q_1 u - p_1 y + \int (q_0 u - p_0 y) \right) \right).$$

Each integral defines a state variable

$$y = q_n u + \underbrace{\int \Bigg(q_{n-1}u - p_{n-1}y + \cdots + \underbrace{\int \Big(q_1 u - p_1 y + \underbrace{\int (q_0 u - p_0 y)}_{x_n} \Big)}_{x_{n-1}} \Bigg)}_{x_1}.$$

Hence we obtain the following (differential) equations:

$$\begin{aligned} y(t) &= q_n u(t) + x_1(t), \\ \dot{x}_1(t) &= q_{n-1}u(t) - p_{n-1}y(t) + x_2(t), \\ \vdots &= \quad \vdots \\ \dot{x}_{n-1}(t) &= q_1 u(t) - p_1 y(t) + x_n(t), \\ \dot{x}_n(t) &= q_0 u(t) - p_0 y(t). \end{aligned}$$

This system corresponds to the state space representation

$$\dot{x}(t) = \begin{bmatrix} -p_{n-1} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 1 & 0 \\ -p_1 & 0 & \cdots & 0 & 1 \\ -p_0 & 0 & \cdots & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} q_{n-1} - p_{n-1}q_n \\ \vdots \\ \vdots \\ q_1 - p_1 q_n \\ q_0 - p_0 q_n \end{bmatrix} u(t), \tag{2.12}$$

$$y(t) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x(t) + q_n u(t). \tag{2.13}$$

We summarize the above result in a theorem.

Theorem 2.1.4. *If $m \leq n$, then the ordinary differential equation* (2.11) *can be written as a state space system with an n-dimensional state space. One possible choice is given by* (2.12) *and* (2.13).

Concluding, every ordinary differential equation with constant coefficients has a space space representation, provided the highest derivative of the input does not exceed the highest derivative of the output.

A natural question is, if the converse holds as well. Hence given a state space representation, can we find an ordinary differential equation relating u and y. We show next that this holds under a mild condition.

Theorem 2.1.5. *Consider the state space system as given in equation* (2.1) *and assume that the output is scalar-valued. If the matrix*

$$\mathfrak{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.14}$$

has full rank, then there exists a differential equation describing the relation of u and y.

Proof. In order to derive the differential equation describing the relation of u and y, we assume that the input u is $(n-1)$-times differentiable. This implies that the output y is also $(n-1)$-times differentiable. As the output is scalar-valued, the matrix $\mathfrak{O}$ is a square matrix. Thus by assumption $\mathfrak{O}$ is invertible. Differentiating the equation $y(t) = Cx(t) + Du(t)$ and using the differential equation for x, we find

$$y^{(1)} = C\,(Ax + Bu) + Du^{(1)}.$$

By induction it is now easy to see that

$$y^{(k)} = CA^k x + \sum_{\ell=0}^{k-1} CA^\ell B u^{(k-1-\ell)} + Du^{(k)}. \tag{2.15}$$

Hence we have that

$$\begin{aligned}\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{n-2}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \\ &= \mathfrak{O}x + \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{n-2}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix}. \end{aligned} \tag{2.16}$$

Since $\mathfrak{O}$ is invertible, x can be expressed by u, y and its derivatives

$$x = \mathfrak{O}^{-1} \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathfrak{O}^{-1} \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{n-2}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix}. \tag{2.17}$$

Inserting this in (2.15) with $k = n$, we find

$$\begin{aligned} y^{(n)} &= CA^n x + \sum_{\ell=0}^{n-1} CA^\ell B u^{(n-1-\ell)} + Du^{(n)} \\ &= CA^n\mathfrak{O}^{-1} \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - CA^n\mathfrak{O}^{-1} \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{n-2}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \\ &\quad + \sum_{\ell=0}^{n-1} CA^\ell B u^{(n-1-\ell)} + Du^{(n)}. \end{aligned} \tag{2.18}$$

This is a differential equation relating u and y, and thus the theorem is proved. □

We apply the above result on the example of the electrical circuit, see Example 2.1.2.

Example 2.1.6. Consider the state space representation of Example 2.1.2. Since $n = 3$, the matrix $\mathfrak{O}$, see (2.14) is given by

$$\mathfrak{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{L_1} \\ -\frac{1}{L_1 C} & -\frac{1}{L_1 C} & 0 \end{bmatrix}$$

which is invertible with inverse

$$\mathfrak{O}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -L_1 C \\ 0 & L_1 & 0 \end{bmatrix}.$$

We calculate the other terms in equation (2.18)

$$CA^3 = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1^2 C} - \frac{1}{L_1 L_2 C} \end{bmatrix}, \; CB = \frac{1}{L_1}, \; CAB = 0, \text{ and } CA^2 B = -\frac{1}{L_1^2 C}.$$

Substituting this in (2.18), we obtain

$$\begin{aligned} y^{(3)} = &\begin{bmatrix} 0 & 0 & -\frac{1}{L_1^2 C} - \frac{1}{L_1 L_2 C} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -L_1 C \\ 0 & L_1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 & -\frac{1}{L_1^2 C} - \frac{1}{L_1 L_2 C} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -L_1 C \\ 0 & L_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{L_1} & 0 \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ u^{(2)} \end{bmatrix} \\ &+ \frac{1}{L_1} u^{(2)} - \frac{1}{L_1^2 C} u. \end{aligned}$$

Evaluating all the products, we find the differential equation

$$\begin{aligned} y^{(3)} &= -\left(\frac{1}{L_1 C} + \frac{1}{L_2 C} \right) y^{(1)} - \left(\frac{1}{L_1^2 C} + \frac{1}{L_1 L_2 C} \right) u + \frac{1}{L_1} u^{(2)} - \frac{1}{L_1^2 C} u \\ &= -\left(\frac{1}{L_1 C} + \frac{1}{L_2 C} \right) y^{(1)} + \frac{1}{L_1} u^{(2)} + \frac{1}{L_1 L_2 C} u. \end{aligned} \tag{2.19}$$

This is exactly equation (1.10).

Hence, we have shown that state space models appear naturally when modeling a system, but they can also be obtained from an ordinary differential equation.

2.2 Solutions of the state space models

In this section, we derive the solution of equation (2.1). This equation consists of a differential equation and an algebraic equation. If the differential equation is solved, then the solution of the algebraic equation follows directly. Thus we concentrate here on the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0. \tag{2.20}$$

In our examples all our matrices and state spaces were real. However, sometimes it is useful to work on $\mathbb{C}^n$ instead of $\mathbb{R}^n$. So we assume that $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$ and $x_0 \in \mathbb{K}^n$, where $\mathbb{K}$ equals either $\mathbb{R}$ or $\mathbb{C}$.

Equation (2.20) is a system of linear inhomogeneous differential equations with constant coefficients. By a *classical solution* of (2.20) we mean a function $x \in C^1([0,\infty);\mathbb{K}^n)$ satisfying (2.20).

Theorem 2.2.1. *If $u \in C([0,\infty);\mathbb{K}^m)$, then the unique classical solution of* (2.20) *is given by the* variation of constant formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds, \qquad t \geq 0. \tag{2.21}$$

For completeness we include a proof of this theorem. However, we assume that the reader is familiar with the fact that the (unique) solution of $\dot{x}(t) = Ax(t), x(0) = x_0$ is given by $e^{At}x_0$.

Proof. We first assume that x is a classical solution of (2.20). Then we have

$$\begin{aligned}
\frac{d}{ds}[e^{A(t-s)}x(s)] &= e^{A(t-s)}\dot{x}(s) - Ae^{A(t-s)}x(s)\\
&= e^{A(t-s)}[Ax(s) + Bu(s)] - Ae^{A(t-s)}x(s)\\
&= e^{A(t-s)}Bu(s),
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^t e^{A(t-s)}Bu(s)\,ds = \int_0^t \frac{d}{ds}[e^{A(t-s)}x(s)]\,ds &= e^{A(t-t)}x(t) - e^{A(t-0)}x(0)\\
&= x(t) - e^{At}x_0.
\end{aligned}$$

Equivalently, equation (2.21) holds. This shows that the solution is unique provided it exists. Thus it remains to show the existence of a classical solution.

Clearly, all we need to show is that x defined by (2.21) is an element of $C^1([0,\infty);\mathbb{K}^n)$ and satisfies the differential equation (2.20). It is easy to see that $x \in C^1([0,\infty);\mathbb{K}^n)$. Moreover, for $t \geq 0$ we have

$$\begin{aligned}
\dot{x}(t) &= Ae^{At}x_0 + Bu(t) + \int_0^t Ae^{A(t-s)}Bu(s)\,ds\\
&= A\left(e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds\right) + Bu(t)\\
&= Ax(t) + Bu(t).
\end{aligned}$$

This concludes the proof of the theorem. □

Thus for continuous input signals, there exists always a unique classical solution of the differential equation (2.20). However, we would like to have the freedom to deal with discontinuous and in particular (square) integrable inputs. Hence we choose as our input function space the set of locally integrable functions, $L^1_{\mathrm{loc}}([0,\infty);\mathbb{K}^m)$. Under mild conditions the existence of a classical solution implies the continuity of the input function u, see Exercise 2.2. Thus for $u \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{K}^m)$ we cannot expect that the differential equation (2.20) possesses a classical solution. However, for these input functions the integral (2.21) is well-defined, and it defines a mild solution as we show in the following.

Definition 2.2.2. A continuous function $x : [0,\infty) \to \mathbb{K}^n$ is called a *mild solution* of (2.20) if x is continuous and satisfies the integrated version of the differential equation (2.20), i.e., if it satisfies

$$x(t) = x_0 + \int_0^t Ax(s) + Bu(s)\,ds \quad \text{for } t \geq 0. \tag{2.22}$$

The following theorem is proved in Exercise 2.2.

Theorem 2.2.3. *Let $u \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{K}^m)$. Then equation* (2.20) *possesses a unique mild solution which is given by* (2.21).

Thus for every $u \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{K}^m)$ the state space system (2.1) has the unique (mild) solution given by

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds, \\ y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)\,ds + Du(t). \end{aligned}$$

In the next session we discuss a special class of state space models.

2.3 Port-Hamiltonian systems

The state space of the state space system (2.1) is given by the Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$, and thus it is natural to work with the standard Euclidean norm/inner product on the state space. However, for many (physical) examples this is not the best choice. For these systems it is preferable to take the energy as the norm. To illustrate this, we return to Example 2.1.2.

Example 2.3.1. The model of the electrical network of Example 2.1.2 is given by, see (2.3)–(2.5),

$$L_1 \frac{dI_{L_1}}{dt}(t) = V_{L_1}(t) = V_C(t) + V(t), \tag{2.23}$$

$$L_2 \frac{dI_{L_2}}{dt}(t) = V_{L_2}(t) = V_C(t), \tag{2.24}$$

$$C \frac{dV_C}{dt}(t) = I_C(t) = -I_{L_1}(t) - I_{L_2}(t). \tag{2.25}$$

As in Example 2.1.2 we choose I_{L_1}, I_{L_2}, and V_C as the state space variables, and so the natural norm on the state space seems to be

$$\left\| \begin{bmatrix} I_{L_1} \\ I_{L_2} \\ V_C \end{bmatrix} \right\| = \sqrt{I_{L_1}^2 + I_{L_2}^2 + V_C^2}.$$

However, this is not the best choice. The preferred norm for this example is the square root of the energy of the (physical) system. For this particular example, the energy equals

$$E = \frac{1}{2}\left(L_1 I_{L_1}^2 + L_2 I_{L_2}^2 + C V_C^2 \right). \tag{2.26}$$

Although this norm is equivalent to the Euclidean norm, the energy norm has some advantages. To illustrate this, we differentiate the energy along solutions. Using (2.23)–(2.25), we find that

$$\begin{aligned} \frac{dE}{dt}(t) &= L_1 \frac{dI_{L_1}}{dt}(t) I_{L_1}(t) + L_2 \frac{dI_{L_2}}{dt}(t) I_{L_2}(t) + C \frac{dV_C}{dt}(t) V_C(t) \\ &= (V_C(t) + V(t)) I_{L_1}(t) + V_C(t) I_{L_2}(t) + (-I_{L_1}(t) - I_{L_2}(t)) V_C(t) \\ &= V(t) I_{L_1}(t). \end{aligned} \tag{2.27}$$

Thus the derivative of the energy along solutions equals the product of the input times the output, see Example 2.1.2. Note that, if we apply no voltage to the system, then the energy will remain constant which implies that the eigenvalues of the matrix A in Example 2.1.2 are on the imaginary axis.

The energy of the system does not only provide a link with physics, also system properties become simpler to prove. The above example is a particular example of a port-Hamiltonian system. Recall that a matrix $\mathcal{H}$ is *positive-definite* if it is self-adjoint and if $x^*\mathcal{H}x > 0$ for all vectors $x \neq 0$. Note that we always write x^* even if we deal with real vectors, and in this situation x^* equals the transpose of the vector.

Definition 2.3.2. Let $\mathcal{H}$ be a positive-definite matrix, and let J be a skew-adjoint matrix, i.e., $J^* = -J$. Then the system

$$\dot{x}(t) = J\mathcal{H}x(t) + Bu(t) \tag{2.28}$$

$$y(t) = B^*\mathcal{H}x(t) \tag{2.29}$$

is called a *port-Hamiltonian system* associated to $\mathcal{H}$ and J. J is called the *structure matrix* and $\mathcal{H}$ is the *Hamiltonian density*. The *Hamiltonian* associated to $\mathcal{H}$ is $\frac{1}{2}x^*\mathcal{H}x$.

This definition is a special case of a much more general definition. We have restricted ourselves here to the linear case, in which the Hamiltonian is quadratic, i.e., equals $\frac{1}{2}x^*\mathcal{H}x$, but other Hamiltonian's are also possible. In many examples the Hamiltonian will be equal to the energy of the system.

Since $\mathcal{H}$ is positive-definite, the expression $\frac{1}{2}x^*\mathcal{H}x$ defines a new norm on $\mathbb{K}^n$, see for example (2.26). This norm is associated to the inner product $\langle x, y\rangle_{\mathcal{H}} := y^*\mathcal{H}x$, and is equivalent to the Euclidean norm. Port-Hamiltonian systems possess the following properties.

Lemma 2.3.3. *Let the norm $\|\cdot\|_{\mathcal{H}}$ on $\mathbb{K}^n$ be defined as $\|x\|_{\mathcal{H}} := \sqrt{\frac{1}{2}x^*\mathcal{H}x}$, where $\mathcal{H}$ is an arbitrary positive-definite matrix, and let x be a (classical) solution of* (2.28)–(2.29). *Then the following equality holds:*

$$\frac{d\|x(t)\|^2_{\mathcal{H}}}{dt} = \operatorname{Re}\left(y(t)^* u(t)\right). \tag{2.30}$$

Proof. As x is a classical solution, we may differentiate the squared norm, and obtain

$$\begin{aligned}
\frac{d\|x(t)\|^2_{\mathcal{H}}}{dt} &= \frac{1}{2}x(t)^*\mathcal{H}\dot{x}(t) + \frac{1}{2}\dot{x}(t)^*\mathcal{H}x(t) \\
&= \frac{1}{2}x(t)^*\mathcal{H}\left(J\mathcal{H}x(t) + Bu(t)\right) + \frac{1}{2}\left(J\mathcal{H}x(t) + Bu(t)\right)^*\mathcal{H}x(t) \\
&= \frac{1}{2}\left(x(t)^*\mathcal{H}J\mathcal{H}x(t) + x(t)^*\mathcal{H}^*J^*\mathcal{H}x(t) + x(t)^*\mathcal{H}Bu(t) + u^*(t)B^*\mathcal{H}x(t)\right) \\
&= 0 + \frac{1}{2}\left(y(t)^*u(t) + u^*(t)y(t)\right),
\end{aligned} \tag{2.31}$$

where we have used that J is skew-adjoint and $\mathcal{H}$ is self-adjoint. □

From the equality (2.30), we can conclude several facts. If $u \equiv 0$, then the Hamiltonian is constant. Thus the solutions lie on isoclines, and the energy remains constant when applying no control. Furthermore, we also get an idea of how to stabilize the system, i.e., how to steer the state to zero. The input $u(t) = -ky(t)$, $k \geq 0$, makes the energy non-increasing. This line of research will be further developed in Chapter 4. We end this chapter by identifying the structure matrix and the Hamiltonian density for the electrical network of Example 2.3.1. If we choose the state x as in Example 2.1.2, then

$$J = \begin{bmatrix} 0 & 0 & \frac{1}{L_1C} \\ 0 & 0 & \frac{1}{L_2C} \\ -\frac{1}{L_1C} & -\frac{1}{L_2C} & 0 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & C \end{bmatrix}. \tag{2.32}$$

However, it is possible to choose a state variable for which J is not depending on the physical parameters, see Exercise 2.3.

2.4 Exercises

2.1. In this exercise we show that a state space representation is not uniquely determined.

(a) Consider Example 2.1.1, and obtain a state space representation of Newton's law with state $x(t) = \begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix}$.

(b) Consider the state space model (2.1). Let T by an invertible $n \times n$-matrix, and define $z(t) = Tx(t)$.
Show that z is also a state of a state space representation, i.e., determine A_T, B_T, C_T, and D_T such that

$$\dot{z}(t) = A_T z(t) + B_T u(t) \qquad y(t) = C_T z(t) + D_T u(t). \tag{2.33}$$

(c) Under the assumption that the output function is scalar-valued and the matrix $\mathfrak{O}$, see (2.14), has full rank, show that the differential equation relating u and y determined by (2.1) equals the differential equation relating u and y determined by (2.33).
Hence although the state is non-unique, the differential equation relating the input and output is.

2.2. In this exercise we study the solutions of the state space representation (2.20) in more detail.

(a) Show that if x is a classical solution of (2.20) and B is injective, then u is continuous.

(b) Prove Theorem 2.2.3.

Beside the notion of a mild solution there is also the notion of a weak solution. A function $x : [0, \infty) \to \mathbb{K}^n$ is said to be a *weak solution* of the differential equation (2.20), if x is continuous and if for every $t > 0$ and every $g \in C^1([0, t]; \mathbb{K}^n)$ the following holds

$$\int_0^t g(\tau)^* \left(Ax(\tau) + Bu(\tau) \right) + \dot{g}(\tau)^* x(\tau) \, d\tau = g(t)^* x(t) - g(0)^* x_0.$$

It can be proved that for every $u \in L^1_{\mathrm{loc}}([0, \infty); \mathbb{K}^m)$, the function x given by (2.21) is the unique weak solution of (2.20).

2.3. In this exercise we investigate again the port-Hamiltonian structure of the electrical network of Example 2.1.2.

(a) Show that the state space model as derived in Example 2.1.2 can be written as a port-Hamiltonian model with the J and $\mathcal{H}$ given by (2.32).

(b) For a port-Hamiltonian system the matrix A in the state space representation is given by $J\mathcal{H}$. However, there are many ways of writing a matrix as a product of two other matrices, and thus the matrices J and $\mathcal{H}$ are not uniquely determined. Therefore, we may add additional conditions. The standard condition is that J may not depend on the physical parameters, which also explains the name "structure matrix". Choose a state x such that the electrical network of Example 2.1.2 can be written as a port-Hamiltonian system with

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Determine the Hamiltonian density in this situation and show that the Hamiltonian remains unchanged.

2.5 Notes and references

After the famous work of Kalman [29] in the 1960s, state space formulation has become one of the most used representation of systems. We refer to [25], [32], and [50] for further results on state space representations. Concerning the results on port-Hamiltonian systems we followed van der Schaft [55].

Chapter 3

Controllability of Finite-Dimensional Systems

In this chapter we study the notion of controllability for finite-dimensional systems as introduced in Chapter 2. For this notion we only need the state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \qquad t \geq 0, \tag{3.1}$$

where A and B are matrices, x_0 is a vector and u is a locally integrable function. As before, we denote by $\mathbb{K}$ the set $\mathbb{R}$ or $\mathbb{C}$ and we denote the system (3.1) by $\Sigma(A, B)$. We recall that the unique mild solution of (3.1) is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds, \qquad t \geq 0. \tag{3.2}$$

Intuitively, the concept of controllability concerns the problem of steering the state of the system from a given state into another state.

3.1 Controllability

There are different notions of controllability available in the literature, most of them are equivalent for finite-dimensional systems. We will see later on, that this is not the case when it comes to infinite-dimensional systems.

Definition 3.1.1. Let $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$. We call the system $\Sigma(A, B)$ *controllable*, if for every $x_0, x_1 \in \mathbb{K}^n$ there exists a $t_1 > 0$ and a function $u \in L^1((0, t_1); \mathbb{K}^m)$ such that the mild solution x of (3.1), given by (3.2), satisfies $x(t_1) = x_1$.

A controllable system has the property that we are able to steer from every point to every other point in the state space. The ability to steer from the origin

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to every point in the state space is known as reachability. This will be defined next. It is clear that controllability implies reachability. In fact, for the system (3.1) these notions are equivalent, see Theorem 3.1.6.

Definition 3.1.2. The system $\Sigma(A,B)$ is *reachable* if for every $x_1 \in \mathbb{K}^n$ there exists a $t_1 > 0$ and a function $u \in L^1((0,t_1);\mathbb{K}^m)$ such that the unique mild solution of (3.1) with $x_0 = 0$ satisfies $x(t_1) = x_1$.

In order to characterize controllability of the system $\Sigma(A,B)$ we introduce the controllability matrix and the controllability Gramian.

Definition 3.1.3. Let $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$. We define the *controllability matrix* $R(A,B)$ by

$$R(A,B) := \begin{bmatrix} B, & AB, & \cdots & ,A^{n-1}B \end{bmatrix}. \tag{3.3}$$

The *controllability Gramian* W_t, $t > 0$, is defined by

$$W_t := \int_0^t e^{As}BB^*e^{A^*s}\,ds. \tag{3.4}$$

It is clear that W_t is an $n\times n$-matrix, whereas $R(A,B)$ is an element of $\mathbb{K}^{n\times nm}$.

For $t > 0$ it is easy to see that the matrix W_t is positive semi-definite, i.e. $x^*W_tx \geq 0$ for all $x \in \mathbb{K}^n$. Moreover, W_t is positive definite if and only if W_t is invertible. In the following we frequently use the *Theorem of Cayley-Hamilton*, which we formulate next. For a proof we refer to standard textbooks on linear algebra.

Theorem 3.1.4 (Theorem of Cayley-Hamilton)**.** *Let $A \in \mathbb{K}^{n\times n}$ with the characteristic polynomial given by $p(\lambda) := \det(\lambda I - A)$. Then A satisfies $p(A) = 0$.*

The Theorem of Cayley-Hamilton states that A^n can be expressed as a linear combination of the lower matrix powers of A. We need some more standard definitions from linear algebra. For a matrix $T \in \mathbb{K}^{p\times q}$, the *rank* of T is denoted by $\operatorname{rk} T$, and the subspace $\operatorname{ran} T$, the *range* of T, is defined by $\operatorname{ran} T := \{y \in \mathbb{K}^p \mid y = Tx$ for some $x \in \mathbb{K}^q\}$.

The range of the controllability Gramian W_t is independent of t and equals the range of the controllability matrix as it is shown in the following proposition.

Proposition 3.1.5. *For every $t > 0$ we have $\operatorname{ran} W_t = \operatorname{ran} R(A,B)$. In particular, W_t is positive definite if and only if $\operatorname{rk} R(A,B) = n$.*

Proof. We show that $(\operatorname{ran} W_t)^\perp = (\operatorname{ran} R(A,B))^\perp$.

Let us first assume that $x \in (\operatorname{ran} R(A,B))^\perp$. Thus $x^*A^kB = 0$ for $k = 0,\ldots,n-1$. By the Theorem of Cayley-Hamilton we obtain

$$x^*A^kB = 0, \qquad k \in \mathbb{N},$$

and in particular

$$x^* e^{As} B = \sum_{k=0}^{\infty} \frac{s^k x^* A^k B}{k!} = 0, \qquad s \geq 0. \tag{3.5}$$

Let $t > 0$ be arbitrary. It follows from (3.5) that

$$x^* W_t = \int_0^t x^* e^{As} B B^* e^{A^* s}\, ds = 0,$$

and therefore $x \in (\operatorname{ran} W_t)^\perp$.

Conversely, let $x \in (\operatorname{ran} W_t)^\perp$ for some $t > 0$. This implies

$$0 = x^* W_t x = \int_0^t \|B^* e^{A^* s} x\|^2 ds.$$

As the function $s \mapsto \|B^* e^{A^* s} x\|^2$ is continuous and non-negative, we obtain $B^* e^{A^* s} x = 0$ for every $s \in [0, t]$. In particular, $x^* B = 0$. Moreover, we obtain

$$0 = \frac{d^k}{ds^k} (B^* e^{A^* s} x)\Big|_{s=0} = B^* (A^*)^k x, \quad k \in \mathbb{N}.$$

This implies $x^* A^k B = 0$ for every $k \in \mathbb{N}$, and thus in particular $x \in (\operatorname{ran} R(A, B))^\perp$. □

We are now in the position to characterize controllability. In particular, we show that if the system $\Sigma(A, B)$ is controllable, then it is controllable in arbitrarily short time. We note, that this result is no longer true for infinite-dimensional systems.

Theorem 3.1.6. *Let $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$ and $t_1 > 0$. Then the following statements are equivalent:*

1. *For every $t_1 > 0$ the system $\Sigma(A, B)$ is* controllable in time t_1, *that is, for every $x_0, x_1 \in \mathbb{K}^n$ there exists a function $u \in L^1((0, t_1); \mathbb{K}^m)$ such that the unique mild solution of* (3.1) *satisfies $x(t_1) = x_1$.*
2. *The system $\Sigma(A, B)$ is controllable.*
3. *The system $\Sigma(A, B)$ is reachable.*
4. *The rank of $R(A, B)$ equals n.*

Proof. Clearly part 1 implies part 2, and part 2 implies part 3.

We now prove that part 3 implies part 4. Therefore, we assume that the system $\Sigma(A, B)$ is reachable. Let $x \in \mathbb{K}^n$ be arbitrary. It is sufficient to show that

$x \in \operatorname{ran} R(A,B)$. As the system $\Sigma(A,B)$ is reachable, there exists a time $t_1 > 0$ and a function $u \in L^1((0,t_1);\mathbb{K}^m)$ such that

$$x = \int_0^{t_1} e^{A(t_1-s)} Bu(s)\, ds.$$

Using Theorem 3.1.4 and the fact that $\operatorname{ran} R(A,B)$ is a (closed) subspace of $\mathbb{K}^n$, we obtain

$$e^{A(t_1-s)}Bu(s) = \sum_{k=0}^{\infty} \underbrace{\frac{(t_1-s)^k}{k!} A^k Bu(s)}_{\in \operatorname{ran} R(A,B)} \in \operatorname{ran} R(A,B).$$

Thus

$$x = \int_0^{t_1} \sum_{k=0}^{\infty} \frac{(t_1-s)^k}{k!} A^k Bu(s)\, ds \in \operatorname{ran} R(A,B),$$

where we have used again the fact that every subspace of a finite-dimensional space is closed. Thus part 4 is proved.

Finally, we prove the implication from part 4 to part 1. Thus we assume that $\operatorname{rk} R(A,B) = n$. Let $x \in \mathbb{K}^n$ and $t_1 > 0$ be arbitrary. We have to show that there is a function $u \in L^1((0,t_1);\mathbb{K}^m)$ such that

$$x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-s)}Bu(s)\, ds.$$

By Proposition 3.1.5, there exists a vector $y \in \mathbb{K}^n$ such that

$$x_1 - e^{At_1}x_0 = W_{t_1} y = \int_0^{t_1} e^{As}BB^* e^{A^*s} y\, ds. \tag{3.6}$$

We define the function $u \in L^1((0,t_1);\mathbb{K}^m)$ by

$$u(s) := B^* e^{A^*(t_1-s)} y, \quad s \in [0,t_1]. \tag{3.7}$$

Using (3.6) and (3.7), we obtain

$$x_1 - e^{At_1}x_0 = \int_0^{t_1} e^{As}BB^*e^{A^*s}y\, ds = \int_0^{t_1} e^{As}Bu(t_1-s)\, ds = \int_0^{t_1} e^{A(t_1-s)}Bu(s)\, ds.$$

In other words, $x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-s)}Bu(s)\, ds$ and thus the theorem is proved. □

Theorem 3.1.6 shows that controllability can be checked by a simple rank condition. Furthermore, we see that if the system is controllable, then the control can be chosen as $u(s) = B^*e^{A^*(t_1-s)}W_{t_1}^{-1}(x_1 - e^{At_1}x_0)$, see (3.6) and (3.7). In particular, the control can be chosen to be smooth, whereas in the definition we only required that the control is integrable.

We close this session with some examples.

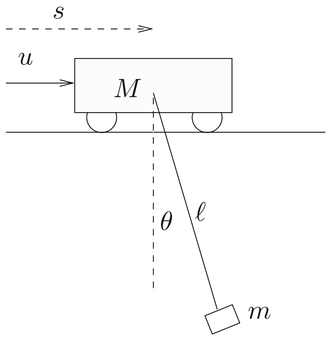

Figure 3.1: Cart with pendulum

Example 3.1.7. We consider a cart on a container bridge to which a pendulum has been attached, see Figure 3.1. The pendulum represents the clamshell. The cart is driven by a motor which at time t exerts a force $u(t)$ taken as control. We assume that all motion occurs in a plane, that is the cart moves along a straight line. Let $M > 0$ be the mass of the cart, $m > 0$ be the mass of the pendulum, which we assume is concentrated at the tip, $\ell > 0$ be the length of the pendulum, s be the displacement of the center of the cart with respect to some fixed point, θ be the angle that the pendulum forms with the vertical, and g be the acceleration of gravity. We assume that the angle θ is small, and thus the kinectic energy equals $\frac{1}{2}M\dot{s}^2 + \frac{1}{2}m(\dot{s} + \ell\dot{\theta})^2$ and the potential energy $\frac{1}{2}mg\ell\theta^2$. The Lagrange's equations give

$$(M+m)\ddot{s} + m\ell\ddot{\theta} = u, \qquad \ddot{s} + \ell\ddot{\theta} + g\theta = 0.$$

Thus we obtain the following linear system of differential equations

$$M\ddot{s} = u + mg\theta, \qquad \ddot{\theta} = -\left(1 + \frac{m}{M}\right)\frac{g}{\ell}\theta - \frac{1}{M\ell}u.$$

Choosing $x := (s, \dot{s}, \theta, \dot{\theta})^T$ as state variable, the corresponding state space representation is given by

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(1+\frac{m}{M})\frac{g}{\ell} & 0 \end{bmatrix}}_{=:A} x(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}}_{=:B} u(t).$$

Thus the controllability matrix $R(A, B)$ of the system is given by

$$R(A,B) = \begin{bmatrix} 0 & \frac{1}{M} & 0 & -\frac{mg}{M^2\ell} \\ \frac{1}{M} & 0 & -\frac{mg}{M^2\ell} & 0 \\ 0 & -\frac{1}{M\ell} & 0 & (1+\frac{m}{M})\frac{g}{M\ell^2} \\ -\frac{1}{M\ell} & 0 & (1+\frac{m}{M})\frac{g}{M\ell^2} & 0 \end{bmatrix}.$$

Dividing the first and the second row by ℓ and adding it to the third and fourth row, respectively, we find that $R(A, B)$ is similar to

$$\begin{bmatrix} 0 & \frac{1}{M} & 0 & -\frac{mg}{M^2\ell} \\ \frac{1}{M} & 0 & -\frac{mg}{M^2\ell} & 0 \\ 0 & 0 & 0 & \frac{g}{M\ell^2} \\ 0 & 0 & \frac{g}{M\ell^2} & 0 \end{bmatrix}.$$

As the rank of this matrix is 4, the rank of $R(A, B)$ is 4, and thus the system is controllable.

Example 3.1.8. Consider the electrical network given by Figure 3.2. Here V denotes the voltage source, L_1, L_2 denote the inductance of the inductors, and C denotes the capacitance of the capacitor. As in Example 1.1.2, using Kirchhoff's laws and

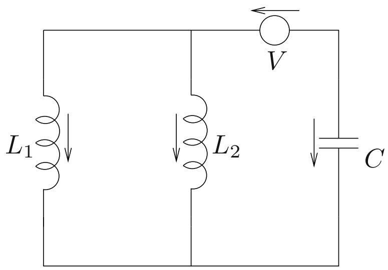

Figure 3.2: Electrical network

the equations $V_L(t) = L\frac{dI_L}{dt}(t)$ and $I_C(t) = C\frac{dV_C}{dt}(t)$, see (1.3) and (1.4), we obtain the system equations

$$L_1 \frac{dI_{L_1}}{dt} = V_{L_1} = V_C - V, \tag{3.8}$$

$$L_2 \frac{dI_{L_2}}{dt} = V_{L_2} = V_C - V, \qquad \text{and} \tag{3.9}$$

$$C \frac{dV_C}{dt} = I_C = -I_{L_1} - I_{L_2}. \tag{3.10}$$

Choosing the state $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} I_{L_1} \\ I_{L_2} \\ V_C \end{pmatrix}$ and the control $u(t) = V(t)$, we receive the state space representation

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & \frac{1}{L_2} \\ -\frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} x(t) + \begin{bmatrix} -\frac{1}{L_1} \\ -\frac{1}{L_2} \\ 0 \end{bmatrix} u(t). \tag{3.11}$$

The controllability matrix is given by

$$R(A,B) = \begin{bmatrix} -\frac{1}{L_1} & 0 & \frac{1}{CL_1^2} + \frac{1}{CL_1L_2} \\ -\frac{1}{L_2} & 0 & \frac{1}{CL_2^2} + \frac{1}{CL_1L_2} \\ 0 & \frac{1}{CL_1} + \frac{1}{CL_2} & 0 \end{bmatrix}. \tag{3.12}$$

Since L_1 times the first row equals L_2 times the second row, we obtain $\operatorname{rk} R(A,B) = 2$ and thus the system $\Sigma(A,B)$ is not controllable. The lack of controllability can also be seen from equation (3.8) and (3.9). The voltage over the inductors is the same independent of the input u. Hence $L_1x_1(t) - L_2x_2(t)$ is constant, and so we cannot choose $x_1(t)$ and $x_2(t)$ independent of each other. Their relation is determined by the initial state.

We have seen in Chapter 2 that the choice for the state is to some extent arbitrary. For example, we can just interchange two state variables. However, the property of controllability is independent of this particular choice of state variables, see Lemma 3.2.2. Thus it is desirable to choose the state such that the matrices A and B have a special form. This question is studied in the next section.

3.2 Normal forms

In the last paragraph of the previous section we discussed the fact that the state is non-unique. Therefore it is often useful to study systems under a change of basis in the state space $\mathbb{K}^n$. Let $T \in \mathbb{K}^{n\times n}$ be invertible. Then the basis transformation $\hat{x} := T^{-1}x$ transforms the system (3.1) into the system, see also Exercise 2.1,

$$\dot{\hat{x}}(t) = T^{-1}AT\hat{x}(t) + T^{-1}Bu(t), \quad \hat{x}(0) = T^{-1}x_0, \qquad t \geq 0. \tag{3.13}$$

This leads to the following definition.

Definition 3.2.1. Let $A, \hat{A} \in \mathbb{K}^{n\times n}$ and $B, \hat{B} \in \mathbb{K}^{n\times m}$. The systems $\Sigma(A,B)$ and $\Sigma(\hat{A},\hat{B})$ are called *similar*, if there exists an invertible matrix $T \in \mathbb{K}^{n\times n}$ such that

$$(\hat{A},\hat{B}) = (T^{-1}AT, T^{-1}B). \tag{3.14}$$

It is easy to see that the controllability matrices of similar systems are related by

$$R(\hat{A},\hat{B}) = T^{-1}R(A,B), \tag{3.15}$$

where (A,B) and $(\hat{A},\hat{B})$ are as in (3.14). This immediately leads to the following result.

Lemma 3.2.2. *Let (A,B) and $(\hat{A},\hat{B})$ be as in* (3.14)*. The system $\Sigma(A,B)$ is controllable if and only if the system $\Sigma(\hat{A},\hat{B})$ is controllable.*

In particular, the statement of the lemma shows that controllability is a property of the system, and not of the particular representation we have chosen for the system. Thus we may try to find a matrix T for which $(\hat{A}, \hat{B})$ has a simple form or equivalently to find a normal form for the system $\Sigma(A, B)$. An application of the following normal form will be given in the next chapter.

Theorem 3.2.3. *Let $A \in \mathbb{K}^{n\times n}$ and $b \in \mathbb{K}^{n\times 1}$ be such that $\Sigma(A, b)$ is controllable. Then there exists an invertible matrix $T \in \mathbb{K}^{n\times n}$ such that*

$$\hat{b} := Tb = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{A} := TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix}.$$

The numbers $a_0, \ldots, a_{n-1} \in \mathbb{K}$ are uniquely determined as the coefficients of the characteristic polynomial of A, i.e., $\det(sI - A) = \sum_{j=0}^{n-1} a_j s^j + s^n$.

Proof. Let $R := R(A, b)$. The assumption that $\Sigma(A, b)$ is controllable implies that the matrix R is invertible. We write

$$R^{-1} = \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix}$$

with $v_i \in \mathbb{K}^n$, $i = 1, \ldots, n$. By the definition of R and v_n^* we have

$$v_n^* A^j b = \begin{cases} 0, & j = 0, \ldots, n-2, \\ 1, & j = n-1. \end{cases} \tag{3.16}$$

Next we show that the vectors $v_n^*, v_n^* A, \ldots, v_n^* A^{n-1}$ are linearly independent. Therefore we assume that there exist scalars $\alpha_1, \ldots, \alpha_n$ such that

$$\sum_{i=1}^{n} \alpha_i v_n^* A^{i-1} = 0. \tag{3.17}$$

Multiplying this equation from the right by b yields

$$\sum_{i=1}^{n} \alpha_i v_n^* A^{i-1} b = 0.$$

Thus (3.16) implies $\alpha_n = 0$. Multiplication of (3.17) by Ab and using again (3.16), then implies $\alpha_{n-1} = 0$. In general, multiplying (3.17) sequently by $b, Ab, A^2b, \ldots$

and using (3.16), implies $\alpha_1 = \cdots = \alpha_n = 0$, that is, the vectors $v_n^*, v_n^* A, \ldots, v_n^* A^{n-1}$ are linearly independent. In particular, this implies that the matrix

$$T := \begin{bmatrix} v_n^* \\ v_n^* A \\ \vdots \\ v_n^* A^{n-1} \end{bmatrix} \in \mathbb{K}^{n \times n} \tag{3.18}$$

is invertible.

By $a_0, \ldots, a_{n-1} \in \mathbb{K}$ we denote the coefficients of the characteristic polynomial of the matrix A, that is,

$$\det(sI - A) = \sum_{j=0}^{n-1} a_j s^j + s^n.$$

The Theorem of Cayley-Hamilton now implies that $A^n = -\sum_{j=0}^{n-1} a_j A^j$ and thus

$$v_n^* A^n = -\sum_{j=0}^{n-1} a_j v_n^* A^j. \tag{3.19}$$

Combining (3.19), (3.18) and (3.16), it is easy to see that

$$TA = \begin{bmatrix} v_n^* A \\ \vdots \\ v_n^* A^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} T \quad \text{and} \quad Tb = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

This proves the assertion of the theorem. □

The previous theorem shows that every controllable system $\Sigma(A, b)$ is similar to a simple system, only containing ones, zeros, and the coefficients of the characteristic polynomial. The following result, known as *Kalman controllability decomposition*, is often useful in order to find simple proofs concerning controllability.

Theorem 3.2.4. *Let $A \in \mathbb{K}^{n \times n}$ and $B \in \mathbb{K}^{n \times m}$. If the system $\Sigma(A, B)$ is not controllable, that is,* $\operatorname{rk} R(A, B) = r < n$, *then $\Sigma(A, B)$ is similar to a system $\Sigma(\hat{A}, \hat{B})$ of the form*

$$(\hat{A}, \hat{B}) = \left(\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right), \tag{3.20}$$

with suitable matrices $A_1 \in \mathbb{K}^{r \times r}, A_2 \in \mathbb{K}^{r \times (n-r)}, A_4 \in \mathbb{K}^{(n-r) \times (n-r)}$ and $B_1 \in \mathbb{K}^{r \times m}$. Moreover, the system $\Sigma(A_1, B_1)$ is controllable.

Proof. Let $Z := \operatorname{ran} R(A,B) = \operatorname{span}\{v_1,\dots,v_r\} \subset \mathbb{K}^n$. Thus we can find vectors $v_{r+1},\dots,v_n \in \mathbb{K}^n$ such that the vectors $v_1,\dots,v_n$ form a basis of $\mathbb{K}^n$. We define

$$T := \begin{bmatrix} v_1, & \dots & , v_n \end{bmatrix} \in \mathbb{K}^{n\times n}.$$

Therefore, T is invertible. We define

$$\hat{B} := T^{-1}B \quad \text{and} \quad \hat{A} := T^{-1}AT.$$

These definitions imply that the systems $\Sigma(A,B)$ and $\Sigma(\hat{A},\hat{B})$ are similar. As $\operatorname{ran} B \subset Z$, we see

$$\hat{B} := T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with $B_1 \in \mathbb{K}^{r\times m}$.

By the Theorem of Cayley-Hamilton we have $\operatorname{ran} AR(A,B) \subset \operatorname{ran} R(A,B) = Z = \operatorname{span}\{v_1,\dots,v_r\}$ and thus

$$AT = T\hat{A} = T\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$$

with suitable matrices $A_1 \in \mathbb{K}^{r\times r}$, $A_2 \in \mathbb{K}^{r\times(n-r)}$ and $A_4 \in \mathbb{K}^{(n-r)\times(n-r)}$. Using the representations of $\hat{A}$ and $\hat{B}$ we obtain

$$R(\hat{A},\hat{B}) = \begin{bmatrix} B_1 & A_1B_1 & \cdots & A_1^{n-1}B_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} = T^{-1}R(A,B).$$

Since multiplication by an invertible matrix does not change the rank, the rank of $R(\hat{A},\hat{B})$ equals $r = \operatorname{rk} R(A,B)$. Furthermore, we obtain

$$\operatorname{rk}\begin{bmatrix} B_1, A_1B_1, \dots, A_1^{n-1}B \end{bmatrix} = r.$$

Since A_1 is an $r \times r$ matrix, the Theorem of Cayley-Hamilton implies that the rank of $\begin{bmatrix} B_1, A_1B_1, \dots, A_1^{n-1}B \end{bmatrix}$ equals the rank of $R(A_1,B_1)$. Thus the rank of $R(A_1,B_1)$ equals the dimension of the associated state space, and thus $\Sigma(A_1,B_1)$ is controllable. □

3.3 Exercises

3.1. In this exercise we obtain some additional tests for controllability of the system $\Sigma(A,B)$ with $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$.

(a) Show that the system $\Sigma(A,B)$ is controllable if and only if for every eigenvector v of A^* we have $v^*B \neq 0$.

(b) Show that the system $\Sigma(A,B)$ is controllable if and only if $\operatorname{rk}[sI - A, B] = n$ for all $s \in \mathbb{C}$.

3.2. Consider the second-order system

$$\ddot{z}(t) = A_0 z(t) + B_0 u(t), \tag{3.21}$$

where z and u are vector-valued functions taking values in $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively. In this exercise we write this as a state space model, and we investigate the controllability of this model in terms of A_0 and B_0.

(a) Use the state vector $x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$ to formulate (3.21) as the state differential equation $\dot{x}(t) = Ax(t) + Bu(t)$.

(b) Prove that the state differential equation found in the previous part is controllable if and only if $R(A_0, B_0)$ has rank n.

3.3. We study a variation of Example 3.1.7, see Figure 3.3. Again we consider a cart on a container bridge, but now we attach two pendulums to the cart, see Figure 3.3. $\ell_1 > 0$ and $\ell_2 > 0$ are the lengths of the pendulums, and we assume that the mass of both pendulums are the same. As in Example 3.1.7

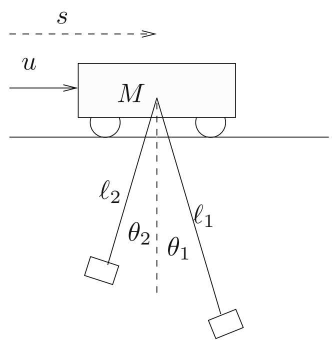

Figure 3.3: Cart with two pendulums

the cart is driven by a motor which at time t exerts a force $u(t)$ taken as control. Neglecting friction and assuming that the angles θ_1 and θ_2 are small, we obtain the following linear system of differential equations

$$\begin{aligned} M\ddot{s} &= u + mg(\theta_1 + \theta_2), \\ \ell_1 \ddot{\theta}_1 &= -\left(1 + \frac{m}{M}\right) g\theta_1 - \frac{mg}{M}\theta_2 - \frac{1}{M}u, \\ \ell_2 \ddot{\theta}_2 &= -\left(1 + \frac{m}{M}\right) g\theta_2 - \frac{mg}{M}\theta_1 - \frac{1}{M}u. \end{aligned}$$

Use Exercise 3.2 to show that the system is controllable if and only if $\ell_1 \neq \ell_2$.

3.4. We continue with the system discussed in Exercise 3.3.

(a) Assume that $\ell_1 = \ell_2$. Find the Kalman controllability decomposition of the system, see Theorem 3.2.4.

(b) Assume that $\ell_1 \neq \ell_2$. Write the state space representation of the system in normal form, see Theorem 3.2.3.

3.4 Notes and references

The results of this chapter can be found for example in [50] and [32]. The characterization of Exercise 3.1.b is known as the Hautus test, see [22].

Chapter 4

Stabilizability of Finite-Dimensional Systems

This chapter is devoted to the stability and stabilizability of state differential equations. Roughly speaking, a system is stable if all solutions converge to zero, and a system is stabilizable if one can find a suitable control function such that the corresponding solution tends to zero. Thus stabilizability is a weaker notion than controllability.

4.1 Stability and stabilizability

We begin by defining stability for the homogeneous state space equation and consider the system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \qquad t \geq 0. \tag{4.1}$$

Here A is an $n \times n$-matrix and x_0 is an n-dimensional vector. As in the previous chapters we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.1.1. Let $A \in \mathbb{K}^{n \times n}$. The differential equation (4.1) is called *exponentially stable*, if for every initial condition $x_0 \in \mathbb{K}^n$ the solution of (4.1) satisfies

$$\lim_{t \to \infty} x(t) = 0.$$

In Chapter 2 we have shown that the unique solution of (4.1) is given by $x(t) = e^{At}x_0$, $t \geq 0$. Using the Jordan normal form for square matrices, the exponential of A can be calculated as $e^{At} = Ve^{\Lambda t}V^{-1}$, where V is an invertible matrix whose columns consist of the generalized eigenvectors of the matrix A, and Λ is a matrix which has non-zero entries only on the diagonal and on the superdiagonal. Moreover, the diagonal elements of the matrix Λ are equal to the

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*,
Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_4,

eigenvalues of A. Thus the differential equation (4.1) is exponentially stable if and only if all eigenvalues of the matrix A lie in the open left half-plane of $\mathbb{C}$. A matrix with the property that all its eigenvalues lie in the left half-plane is called a *Hurwitz matrix*. In the following we denote by $\sigma(A)$ the set of all eigenvalues of the matrix A.

From the representation of the exponential of a matrix, the following result follows immediately.

Lemma 4.1.2. *The differential equation* (4.1) *is exponentially stable if and only if there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$\|e^{At}x_0\| \leq Me^{-\omega t}\|x_0\|, \qquad t \geq 0, x_0 \in \mathbb{K}^n.$$

Thus, if all solutions of (4.1) converge to zero for $t \to \infty$, then all solutions converge uniformly and exponentially to zero. This motivates the notion "exponentially stable".

If the matrix A possesses eigenvalues with non-negative real part, then there are solutions of equation (4.1) which do not tend to zero. It is thus desirable to stabilize the system, that is, we study again a system with an input

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \qquad t \geq 0, \tag{4.2}$$

and we try to find a suitable input function u such that the corresponding solution x converges to zero for $t \to \infty$.

Definition 4.1.3. Let $A \in \mathbb{K}^{n\times n}$ and $B \in \mathbb{K}^{n\times m}$. We call the system $\Sigma(A, B)$ *stabilizable*, if for every $x_0 \in \mathbb{K}^n$ there exists a function $u \in L^1_{\text{loc}}((0,\infty); \mathbb{K}^m)$ such that the unique solution of (4.2) converges to zero for $t \to \infty$.

Clearly, controllable systems $\Sigma(A, B)$ are stabilizable. We only have to steer the state to zero in some time t_1 and to choose the input u to be zero on the interval $[t_1, \infty)$, see Exercise 4.1.

4.2 The pole placement problem

It is the aim of the following two sections to characterize stabilizability and to show that the stabilizing control function u can be obtained via a feedback law $u(t) = Fx(t)$. In order to show the existence of such a feedback, we first have to study the more general problem of *pole placement*:

Pole placement problem. Given $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$ and complex numbers $\lambda_1, \ldots, \lambda_n$ the question is whether there exists a matrix $F \in \mathbb{K}^{m\times n}$ such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix $A + BF$.

If the pole placement problem is solvable, then we can move the eigenvalues of closed loop system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad u(t) = Fx(t), \qquad t \geq 0, \tag{4.3}$$

to arbitrary points in $\mathbb{C}$. Note that equations (4.3) are equivalent to

$$\dot{x}(t) = (A + BF)x(t) \qquad t \geq 0.$$

So far all results hold for real- and complex-valued matrices. However, the situation is different for the pole placement problem. For example, the points $\lambda_1, \dots, \lambda_n$ are always allowed to be complex. To simplify the exposition we formulate results for the complex situation only. In the real situation the results hold with obvious modifications.

The following theorem shows that the system $\Sigma(A, B)$ is controllable if the pole placement problem is solvable.

Theorem 4.2.1. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then* 1. *implies* 2.*, and* 2. *implies* 3.*, where*

1. *The pole placement problem is solvable.*
2. *There exists an $F \in \mathbb{C}^{m\times n}$ such that the eigenvalues of $A + BF$ are all different from the eigenvalues of A, i.e., $\sigma(A + BF) \cap \sigma(A) = \emptyset$.*
3. *The system $\Sigma(A, B)$ is controllable.*

We note that the three statements of Theorem 4.2.1 are equivalent, see Corollary 4.2.6.

Proof. Clearly 1. implies 2. Thus we concentrate on the implication from 2. to 3.

If $\Sigma(A, B)$ is not controllable, then $\operatorname{rk} R(A, B) =: r < n$. By Theorem 3.2.4 there exists an invertible T such that

$$(T^{-1}AT, T^{-1}B) := \left(\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right)$$

with suitable matrices $A_1 \in \mathbb{C}^{r\times r}, A_2 \in \mathbb{C}^{r\times(n-r)}, A_4 \in \mathbb{C}^{(n-r)\times(n-r)}$ and $B_1 \in \mathbb{C}^{r\times m}$. If we write an arbitrary matrix $F \in \mathbb{C}^{m\times n}$ in the form $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} T^{-1}$ with $F_1 \in \mathbb{C}^{m\times r}$, we obtain

$$A + BF = T \begin{bmatrix} A_1 + B_1F_1 & A_2 + B_1F_2 \\ 0 & A_4 \end{bmatrix} T^{-1}.$$

As the eigenvalue of A_4 are fixed, the intersection of the eigenvalues of $A+BF$ and A always contains the eigenvalues of A_4. Thus for every F this intersection is non empty, which contradicts the assertion in 2. Concluding, $\Sigma(A, B)$ is controllable. □

In the following we show, that controllability is actually equivalent to the solvability of the pole placement problem.

Theorem 4.2.2. *Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. If the system $\Sigma(A, b)$ is controllable, then there exists a matrix $f \in \mathbb{C}^{1\times n}$ such that the matrix $A + bf$ has the eigenvalues $\lambda_1, \dots, \lambda_n$, counted with multiplicities.*

Proof. By Theorem 3.2.3 there exists an invertible matrix $T \in \mathbb{C}^{n\times n}$ such that

$$\hat{b} := Tb = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \hat{A} := TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \tag{4.4}$$

and

$$\det(sI - A) = \sum_{j=0}^{n-1} a_j s^j + s^n.$$

We define the polynomial p by

$$p(s) = \prod_{j=1}^{n} (s - \lambda_j) = s^n + \sum_{j=0}^{n-1} p_j s^j, \tag{4.5}$$

and the matrices $\hat{f}, f \in \mathbb{C}^{1\times n}$ by

$$\hat{f} := \begin{bmatrix} a_0 - p_0, & \dots & , a_{n-1} - p_{n-1} \end{bmatrix} \quad \text{and} \quad f = \hat{f}T. \tag{4.6}$$

Thus $A + bf = T^{-1}(\hat{A} + \hat{b}\hat{f})T$, which implies that the set of eigenvalues of $A + bf$ equals the set of eigenvalues of $\hat{A} + \hat{b}\hat{f}$, counted with multiplicities. Moreover, from (4.4) and (4.6) we obtain that

$$\hat{A} + \hat{b}\hat{f} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix}.$$

Thus the characteristic polynomial of $\hat{A} + \hat{b}\hat{f}$ equals p. By (4.5) we conclude that $\hat{A} + \hat{b}\hat{f}$ and thus $A + bf$ has exactly the eigenvalues $\lambda_1, \dots, \lambda_n$, again counted with multiplicities. □

Proposition 4.2.3. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ be such that $\Sigma(A, B)$ is controllable and let $b \in \mathbb{C}^n \setminus \{0\}$. Then there exists a matrix $F \in \mathbb{C}^{m\times n}$ such that the system $\Sigma(A + BF, b)$ is controllable.*

Proof. We define $r := \operatorname{rk} R(A, b) \le n$. If $r = n$, then the system $\Sigma(A, b)$ is controllable and thus we can choose $F = 0$. Let $r < n$. By Theorem 3.2.4, we can assume without loss of generality that

$$(A, b) = \left(\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}, \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \right)$$

with suitable matrices $A_1 \in \mathbb{C}^{r\times r}, A_2 \in \mathbb{C}^{r\times(n-r)}, A_4 \in \mathbb{C}^{(n-r)\times(n-r)}$, $b_1 \in \mathbb{C}^r$, and $\Sigma(A_1, b_1)$ is controllable. Since $\Sigma(A_1, b_1)$ is controllable, the square matrix

$$[b_1, A_1b_1, \cdots, A_1^{r-1}b_1]$$

has rank r. Thus by the Theorem of Cayley-Hamilton

$$r = \operatorname{rk}\begin{bmatrix} b_1 & A_1b_1 & \cdots & A_1^{r-1}b_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \operatorname{rk}\begin{bmatrix} b_1 & A_1b_1 & \cdots & A_1^{n-1}b_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \operatorname{rk} R(A, b). \tag{4.7}$$

This implies that the vectors $v_i := A^{i-1}b$, $i = 1, \ldots, r$, are linearly independent and

$$\operatorname{span}\{v_1, \ldots, v_r\} = \operatorname{ran} R(A, b).$$

As the system $\Sigma(A, B)$ is controllable, we have $\operatorname{rk} R(A, B) = n$. Next we want to show that there exists a vector $\hat{b} \in \operatorname{ran} B\backslash\operatorname{ran} R(A, b)$. We assume that this does not hold, that is, we assume that $\operatorname{ran} B \subset \operatorname{ran} R(A, b)$. Then the Theorem of Cayley-Hamilton implies $\operatorname{ran} R(A, B) \subset \operatorname{ran} R(A, b)$, which is in contradiction to $\operatorname{rk} R(A, b) = r < n = \operatorname{rk} R(A, B)$.

Let $\hat{b} \in \operatorname{ran} B\backslash\operatorname{ran} R(A, b)$ and $u \in \mathbb{C}^m$ with $Bu = \hat{b}$. We define $v_{r+1} = Av_r + \hat{b}$. Since by (4.7) $Av_r \in \operatorname{ran} R(A, b) = \operatorname{span}\{v_1, \ldots, v_r\}$, this choice for v_{r+1} implies that the vectors $v_1, \ldots, v_{r+1}$ are linearly independent. We choose a matrix $F \in \mathbb{C}^{n\times n}$ with the property

$$Fv_i = \begin{cases} 0, & i = 1, \ldots, r-1, \\ u, & i = r, \end{cases}$$

and we define

$$F_1 := A + BF.$$

Then we have

$$F_1^{i-1}b = v_i, \qquad i = 1, \ldots, r+1,$$

and thus $\operatorname{rk} R(F_1, b) \geq r + 1$. If $\operatorname{rk} R(F_1, b) = n$, then the statement of the proposition follows. If $\operatorname{rk} R(F_1, b) < n$, then we first show that $\Sigma(F_1, B)$ is controllable. By Theorem 3.1.6 it is sufficient to prove that $\Sigma(F_1, B)$ is reachable. Let $x_1 \in \mathbb{C}^n$ be arbitrarily. Since $\Sigma(A, B)$ is reachable, there is some $t_1 > 0$ and an input $u \in L^1((0, t_1); \mathbb{C}^m)$ such that the mild solution of $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = 0$, satisfies $x(t_1) = x_1$. Now we choose $v = u - Fx$ as input for the system $\Sigma(F_1, B)$ and the mild solution of $\dot{x}(t) = F_1x(t) + Bv(t)$, $x(0) = 0$, satisfies $x(t_1) = x_1$, which shows that $\Sigma(F_1, B)$ is reachable. As $\Sigma(F_1, B)$ is controllable as well, we can apply the above procedure to the system $\Sigma(F_1, b)$, until the rank of the matrix $R(F_j, b)$ equals n. □

Next we are in the position to show that controllability implies the solvability of the pole placement problem.

Theorem 4.2.4. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$ be such that the system $\Sigma(A,B)$ is controllable and let $\lambda_1,\dots,\lambda_n \in \mathbb{C}$. Then there exists a matrix $F \in \mathbb{C}^{m\times n}$ such that the matrix $A+BF$ has exactly the eigenvalues $\lambda_1,\dots,\lambda_n$, counted with multiplicities.*

Proof. We choose a vector $b = Bu \in \operatorname{ran} B\backslash\{0\}$. By Proposition 4.2.3 there exists a matrix $\hat{F} \in \mathbb{C}^{m\times n}$ such that the system $\Sigma(A+B\hat{F}, b)$ is controllable. Finally, by Theorem 4.2.2 there exists a vector f such that the matrix $A + B\hat{F} + bf$ has exactly the eigenvalues $\lambda_1,\dots,\lambda_n$, counted with multiplicities. Thus, by the choice of $F := \hat{F} + uf$ the statement of the theorem is proved. □

As a consequence of the theorem, we obtain that for every controllable system $\Sigma(A,B)$ there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $A+BF$ is a Hurwitz matrix.

Corollary 4.2.5. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. If the system $\Sigma(A,B)$ is controllable, then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $A+BF$ is a Hurwitz matrix.*

Moreover, we have shown the equivalence of the pole placement problem and controllability.

Corollary 4.2.6. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

1. *The pole placement problem is solvable.*
2. *There exists an $F \in \mathbb{C}^{m\times n}$ such that the eigenvalues of $A+BF$ are all different from the eigenvalues of A, i.e., $\sigma(A+BF)\cap\sigma(A) = \emptyset$.*
3. *The system $\Sigma(A,B)$ is controllable.*

If we want to place the poles arbitrarily, then we need controllability. However, if one only wants to stabilize the system, then a weaker condition suffices as shown in the following section.

4.3 Characterization of stabilizability

In Definition 4.1.3 we have defined stabilizability as an open loop notion, i.e., there exists an input such that the corresponding state converges to zero. In this section we show that if it is possible to stabilize the system via a well-chosen input, then it is also possible via state feedback, i.e., via an input of the form $u(t) = Fx(t)$. The following theorem formulates this assertion.

Theorem 4.3.1. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

1. *The system $\Sigma(A,B)$ is stabilizable.*
2. *There exists a matrix $F \in \mathbb{C}^{m\times n}$ such that $A+BF$ is a Hurwitz matrix.*

3. *For every eigenvector* v *of* A^* *which belongs to an eigenvalue* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq 0$ *we have:* $v^* B \neq 0$.

Proof. Clearly, part 2. implies part 1.

Next we show that part 1. implies part 3. In order to prove this implication we assume that the system $\Sigma(A, B)$ is stabilizable and that there exists an eigenvector v of A^* which belongs to an eigenvalue $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ such that $v^* B = 0$. It then follows that

$$v^* A^n B = ((A^n)^* v)^* B = \bar{\lambda}^n v^* B = 0, \quad n \in \mathbb{N},$$

and thus

$$v^* e^{At} B = 0, \qquad t \geq 0. \tag{4.8}$$

As the system $\Sigma(A, B)$ is stabilizable, there exists a function $u \in L^1_{\text{loc}}((0, \infty); \mathbb{C}^m)$ such that the function

$$x(t) := e^{At} v + \int_0^t e^{A(t-s)} B u(s)\, ds, \quad t \geq 0, \tag{4.9}$$

converges to zero for $t \to \infty$. Multiplying equation (4.9) from the left by v^* we obtain

$$v^* x(t) = v^* e^{At} v + \int_0^t v^* e^{A(t-s)} B u(s)\, ds = v^* e^{\bar{\lambda} t} v + 0 = e^{\bar{\lambda} t} \|v\|^2, \quad t \geq 0.$$

The left-hand side converges to zero for $t \to \infty$, whereas the right-hand side does not converge to zero. This leads to a contradiction. Thus part 1. implies part 3.

It remains to show that part 3. implies part 2. If the system $\Sigma(A, B)$ is controllable, then the statement follows from Corollary 4.2.5. If the system $\Sigma(A, B)$ is not controllable, then $\operatorname{rk} R(A, B) =: r < n$. By Theorem 3.2.4 there exists an invertible square matrix T such that

$$(A, B) = (T^{-1} \hat{A} T, T^{-1} \hat{B}) = \left(T^{-1} \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} T, T^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right), \tag{4.10}$$

where A_4 is a $(n - r) \times (n - r)$-matrix and the system $\Sigma(A_1, B_1)$ is controllable. Next we show that all eigenvalues of A_4 lie in the open left half plane. Let α be an eigenvalue of A_4^* and let v_2 be a corresponding eigenvector. Using (4.10) it is easy to see that $v := T^* \left[\begin{smallmatrix} 0 \\ v_2 \end{smallmatrix} \right] \in \mathbb{C}^n$ is an eigenvector of A^* with respect to the eigenvalue α. This implies that

$$v^* B = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}^* T T^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & v_2^* \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = 0.$$

By assumption this can only happen if $\operatorname{Re}\alpha < 0$. This shows that A_4 is a Hurwitz matrix. As the system $\Sigma(A_1, B_1)$ is controllable, by Corollary 4.2.5 there exists a matrix $F_1 \in \mathbb{C}^{m\times r}$ such that $A_1 + B_1F_1$ is a Hurwitz matrix. Choosing the feedback $F := \begin{bmatrix} F_1 & 0 \end{bmatrix} T$, we obtain that

$$A + BF = T^{-1}\begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} T + T^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}\begin{bmatrix} F_1 & 0 \end{bmatrix} T = T^{-1}\begin{bmatrix} A_1 + B_1F_1 & A_2 \\ 0 & A_4 \end{bmatrix} T.$$

Since $\left[\begin{smallmatrix} A_1+B_1F_1 & A_2 \\ 0 & A_4 \end{smallmatrix}\right]$ is a Hurwitz matrix, the matrix $A + BF$ is Hurwitz as well. Thus part 3. implies part 2. □

Summarizing, we have shown that a stabilizable system can be stabilized by state feedback. However, we still lack a simple condition which characterizes stabilizability. This will be given next, for which we need the following concept.

Definition 4.3.2. Let $A \in \mathbb{K}^{n\times n}$. We call a subspace $Z \subset \mathbb{K}^n$ *A-invariant*, if $AZ \subset Z$.

Theorem 4.3.3. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

1. *The system $\Sigma(A, B)$ is stabilizable.*
2. *There exist two A-invariant subspaces X_s and X_u of $\mathbb{C}^n$ such that*
   a) $\mathbb{C}^n = X_s \oplus X_u$.
   b) *The system $\Sigma(A, B)$ is similar to*
   $$\Sigma\left(\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \begin{bmatrix} B_s \\ B_u \end{bmatrix}\right).$$
   c) *The matrix A_s is a Hurwitz matrix.*
   d) *The system $\Sigma(A_u, B_u)$ is controllable.*

Proof. We first prove the implication 2. ⇒ 1. Clearly, the system $\Sigma(A_s, B_s)$ is stabilizable by a matrix $F = 0$, and the system $\Sigma(A_u, B_u)$ is stabilizable by Corollary 4.2.5. Thus it is easy to see that the system $\Sigma(A, B)$ is stabilizable.

Next we assume that $\Sigma(A, B)$ is stabilizable. We now define X_s by the span of all generalized eigenspaces corresponding to eigenvalues of A with negative real part and X_u by the span of all generalized eigenspaces corresponding to eigenvalues of A with non-negative real part. Thus, X_s and X_u are A-invariant, $X_s \cap X_u = \{0\}$ and $X_s \oplus X_u = \mathbb{C}^n$. If we choose our basis of $\mathbb{C}^n$ accordingly, it is easy to see that the system $\Sigma(A, B)$ is similar to a system of the form

$$\Sigma\left(\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \begin{bmatrix} B_s \\ B_u \end{bmatrix}\right).$$

By definition of the space X_s, the matrix A_s is a Hurwitz matrix.

It remains to show that $\Sigma(A_u, B_u)$ is controllable. We know that the system $\Sigma(A_u, B_u)$ is stabilizable and that all eigenvalues of A_u lie in the closed right half plane. Thus by Corollary 4.2.6 the system $\Sigma(A_u, B_u)$ is controllable. □

4.4 Stabilization of port-Hamiltonian systems

In this section we return to the class of (finite-dimensional) port-Hamiltonian systems as introduced in Section 2.3. We show that port-Hamiltonian systems are stabilizable if and only if they are controllable. Furthermore, if such a system is stabilizable, then the system is stabilizable by (static) output feedback, i.e., by choosing the input $u = -ky$, with $k > 0$.

We consider the port-Hamiltonian system from Section 2.3 which is given by

$$\dot{x}(t) = J\mathcal{H}x(t) + Bu(t), \tag{4.11}$$

$$y(t) = B^*\mathcal{H}x(t), \tag{4.12}$$

where J is skew-adjoint, i.e., $J^* = -J$ and $\mathcal{H}$ is a positive-definite matrix.

Theorem 4.4.1. *The system* (4.11)–(4.12) *is stabilizable if and only if it is controllable. Furthermore, it is controllable if and only if it is stabilizable by applying the feedback* $u(t) = -ky(t) = -kB^*\mathcal{H}x(t)$ *with* $k > 0$*.*

Proof. We define a new norm $\|\cdot\|_H$ on $\mathbb{C}^n$ by

$$\|x\|_H^2 := \frac{1}{2}x^*\mathcal{H}x, \tag{4.13}$$

which is equivalent to the Euclidean norm, and along solutions we have

$$\frac{d\|x(t)\|_H^2}{dt} = \mathrm{Re}\left(u(t)^*y(t)\right). \tag{4.14}$$

For $u \equiv 0$ and for any initial condition $x(0)$, this implies

$$\|x(t)\|_H = \|x(0)\|_H. \tag{4.15}$$

Thus all eigenvalues of the matrix $J\mathcal{H}$ lie on the imaginary axis. By Theorem 4.3.3 we conclude that the system is stabilizable if and only if it controllable. It remains to construct a stabilizing controller.

So we assume that the system is controllable, and as a candidate for a stabilizing controller we choose $u(t) = -ky(t) = -kB^*\mathcal{H}x(t)$ with $k > 0$. Using (4.14), we have

$$\frac{d\|x(t)\|_H^2}{dt} = -k\|y(t)\|^2. \tag{4.16}$$

Applying the feedback $u(t) = -kB^*\mathcal{H}x(t)$, we obtain the state equation

$$\dot{x}(t) = \left(J\mathcal{H} - kBB^*\mathcal{H}\right)x(t). \tag{4.17}$$

Equation (4.16) shows that for every initial condition the function $t \mapsto \|x(t)\|_H^2$ is non-increasing, and this implies that all eigenvalues of $J\mathcal{H} - kBB^*\mathcal{H}$ lie in the closed left half-plane. Hence we can conclude stability if there are no eigenvalues on the imaginary axis.

Let $\lambda \in i\mathbb{R}$ be a purely imaginary eigenvalue of $J\mathcal{H}-kBB^*\mathcal{H}$ with eigenvector $v \in \mathbb{C}^n$, i.e.,

$$(J\mathcal{H} - kBB^*\mathcal{H})v = \lambda v. \tag{4.18}$$

Multiplying this equation from the left by $v^*\mathcal{H}$ gives

$$v^*\mathcal{H}J\mathcal{H}v - kv^*\mathcal{H}BB^*\mathcal{H}v = \lambda v^*\mathcal{H}v. \tag{4.19}$$

Multiplying the Hermitian conjugate of equation (4.18) from the right by $\mathcal{H}v$, and using the fact that $\mathcal{H}$ is self-adjoint and J is skew-adjoint, we obtain

$$-v^*\mathcal{H}J\mathcal{H}v - kv^*\mathcal{H}BB^*\mathcal{H}v = -\lambda v^*\mathcal{H}v.$$

Adding this equation to equation (4.18), we find that $kv^*\mathcal{H}BB^*\mathcal{H}v = 0$. Since $k > 0$, this is equivalent to

$$B^*\mathcal{H}v = 0. \tag{4.20}$$

Equation (4.18) now implies that

$$J\mathcal{H}v = \lambda v.$$

Taking the Hermitian conjugate, we find

$$\lambda^* v^* = -\lambda v^* = v^*\mathcal{H}^*J^* = -v^*\mathcal{H}J. \tag{4.21}$$

Combining (4.20) and (4.21), we see that

$$v^*\mathcal{H}J\mathcal{H}B = -\lambda^* v^*\mathcal{H}B = \lambda v^*\mathcal{H}B = 0.$$

Similarly, we find for $k \in \mathbb{N}$,

$$v^*\mathcal{H}\left(J\mathcal{H}\right)^k B = 0, \tag{4.22}$$

which implies

$$v^*\mathcal{H}R(J\mathcal{H}, B) = v^*\mathcal{H}\left(B, J\mathcal{H}B, \ldots, \left(J\mathcal{H}\right)^{n-1} B\right) = 0.$$

As the system $\Sigma(J\mathcal{H}, B)$ is controllable, we obtain $v^*\mathcal{H} = 0$, or equivalently, $v = 0$. This is in contradiction to the fact that v is an eigenvector.

Summarizing, if the system is controllable, then the closed loop system matrix $J\mathcal{H} - kBB^*\mathcal{H}$ has no eigenvalues in the closed right half-plane, and so we have stabilized the system. □

4.5 Exercises

4.1. Assume that the system $\Sigma(A, B)$ is controllable. Show that $\Sigma(A, B)$ is also stabilizable by steering the state to zero in some time t_1 and choosing the input u to be zero on the interval $[t_1, \infty)$.

4.2. Show that the system $\Sigma(A, B)$ is stabilizable if and only if

$$\operatorname{rk}\begin{bmatrix} sI - A & B \end{bmatrix} = n$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$.

4.3. In this exercise we study the system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -2 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -2 \end{bmatrix}}_{=:A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{=:B} u(t).$$

(a) Show that the system $\Sigma(A, B)$ is not controllable.

(b) Show that the system $\Sigma(A, B)$ is stabilizable.

(c) Find a stabilizing feedback control for this system.

4.4. Consider the following matrix

$$A = \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 0 & 1 & 7 \\ -4 & 0 & 0 & -1 \\ 0 & -5 & -1 & 0 \end{bmatrix}. \tag{4.23}$$

(a) Find a skew-adjoint J and a positive-definite matrix $\mathcal{H}$ such that $A = J\mathcal{H}$.
Hint: Assume $J^* = J^{-1}$ and calculate A^*A.

(b) Now we choose $b = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. Show that $\Sigma(A, b)$ is controllable.

(c) Design a stabilizing controller.

4.6 Notes and references

Since stabilizability is one of the important problem within systems theory, its solution can be found in any book on finite-dimensional systems theory. We refer to our standard references [50] and [32]. The characterization of stabilizability for port-Hamiltonian systems can be found in [55]. The characterization of stabilizability of Exercise 4.2 is known as the Hautus test for stabilizability, see also Exercise 3.1.b.

Chapter 5

Strongly Continuous Semigroups

In Chapter 2 we showed that the examples of Chapter 1, which were described by ordinary differential equations, can be written as a first-order differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) = Du(t), \tag{5.1}$$

where $x(t)$ is a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$. In Chapters 5 and 6 we show how systems described by partial differential equations can be written in the same form by using an infinite-dimensional state space. The formulation of the inputs and outputs is postponed till Chapter 10. Note that for partial differential equations the question of existence and uniqueness of solutions is more difficult than for ordinary differential equations. Thus we focus first on homogeneous partial differential equations. We begin by introducing the solution operator, and only in Section 5.3 do we show how to rewrite a p.d.e. as an abstract differential equation $\dot{x}(t) = Ax(t)$.

5.1 Strongly continuous semigroups

We start with a simple example.

Example 5.1.1. Consider a metal bar of length 1 that is insulated at the boundary:

$$\begin{aligned}
&\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) \qquad x(\zeta,0) = x_0(\zeta),\\
&\frac{\partial x}{\partial \zeta}(0,t) = 0 = \frac{\partial x}{\partial \zeta}(1,t).
\end{aligned} \tag{5.2}$$

$x(\zeta,t)$ represents the temperature at position $\zeta \in [0,1]$ at time $t \geq 0$ and x_0 represents the initial temperature profile. The two boundary conditions state that

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there is no heat flow at the boundary, see Example 1.1.5, and thus the bar is insulated.

In order to calculate the solution of (5.2) we try to find a solution of the form $x(\zeta,t) = f(t)g(\zeta)$. Substituting this equation in (5.2) and using the boundary conditions we obtain

$$f(t)g(\zeta) = \alpha_n e^{-n^2\pi^2 t}\cos(n\pi\zeta), \tag{5.3}$$

where $\alpha_n \in \mathbb{R}$ or $\mathbb{C}$, and $n \in \mathbb{N}$. This solution satisfies the p.d.e. and the boundary conditions, but most likely not the initial condition. By the linearity of the p.d.e. (5.2) it is easy to see that

$$x_N(\zeta,t) = \sum_{n=0}^{N} \alpha_n e^{-n^2\pi^2 t}\cos(n\pi\zeta) \tag{5.4}$$

satisfies the p.d.e. and the boundary conditions as well. The corresponding initial condition $x_N(\zeta,0) = \sum_{n=0}^{N} a_n \cos(n\pi\zeta)$ is a Fourier polynomial. Note that every function q in $L^2(0,1)$ can be represented by its Fourier series, i.e.,

$$q(\cdot) = \sum_{n=0}^{\infty} \alpha_n \cos(n\pi\cdot), \tag{5.5}$$

with equality/convergence in $L^2(0,1)$, $\alpha_0 = \int_0^1 q(\zeta)\,d\zeta$ and

$$\alpha_n = 2\int_0^1 q(\zeta)\cos(n\pi\zeta)\,d\zeta, \qquad n = 1,2,\ldots.$$

If $x_0 \in L^2(0,1)$, then we can define $\alpha_n, n \in \mathbb{N}\cup\{0\}$ as the corresponding Fourier coefficients and

$$x(\zeta,t) := \sum_{n=0}^{\infty} \alpha_n e^{-n^2\pi^2 t}\cos(n\pi\zeta). \tag{5.6}$$

Since for $t \geq 0$ we have $e^{-n^2\pi^2 t} \leq 1$, the function $x(\cdot,t)$ is an element of $L^2(0,1)$. By construction, the initial condition is satisfied as well. However, as interchanging differentiation and (infinite) summation is not always allowed, it is unclear if this function satisfies the p.d.e. (5.2). Nevertheless the mapping $x_0 \mapsto x(\cdot,t)$ defines an operator, which would assign to an initial condition its corresponding solution at time t, provided x is the solution.

This example motivates the necessity for generalizing the concept of "e^{At}" on abstract spaces and the necessity for clarifying the concept of "solution" of differential equations on abstract spaces. The answer is, of course, the well-known semigroup theory that we develop here for the special case of strongly continuous semigroups on a Hilbert space.

We denote by X a real or complex (separable) Hilbert space, with inner product $\langle\cdot,\cdot\rangle_X$ and norm $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle_X}$. By $\mathcal{L}(X)$ we denote the class of linear bounded operators from X to X.

Definition 5.1.2. Let X be a Hilbert space. $(T(t))_{t\geq 0}$ is called a *strongly continuous semigroup* (or short C_0-*semigroup*) if the following holds:

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X, i.e., $T(t) \in \mathcal{L}(X)$;
2. $T(0) = I$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$, i.e., $t \mapsto T(t)$ is strongly continuous at zero.

We call X the *state space*, and its elements *states*. The easiest example of a strongly continuous semigroup is the exponential of a matrix. That is, let A be an $n \times n$ matrix, the matrix-valued function $T(t) = e^{At}$ defines a C_0-semigroup on the Hilbert space $\mathbb{R}^n$, see Exercise 5.1. Another example is presented next.

Example 5.1.3. Let $\{\phi_n,\ n \geq 1\}$ be an orthonormal basis of the separable Hilbert space X, and let $\{\lambda_n,\ n \geq 1\}$ be a sequence of complex numbers. Then

$$T(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x, \phi_n \rangle \phi_n \tag{5.7}$$

is a bounded, linear operator if and only if $\{e^{\operatorname{Re}\lambda_n t}, n \geq 1\}$ is a bounded sequence in $\mathbb{R}$, and this is the case for $t > 0$ if and only if

$$\sup_{n\geq 1} \operatorname{Re}\lambda_n < \infty.$$

Under this assumption, we have

$$\|T(t)\| \leq e^{\omega t} \tag{5.8}$$

with $\omega = \sup_{n\geq 1} \operatorname{Re}\lambda_n$. Furthermore,

$$T(t+s)x = \sum_{n=1}^{\infty} e^{\lambda_n (t+s)} \langle x, \phi_n \rangle \phi_n$$

and

$$\begin{aligned} T(t)T(s)x &= \sum_{n=1}^{\infty} e^{\lambda_n t} \langle T(s)x, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} e^{\lambda_n t} \left\langle \sum_{m=1}^{\infty} e^{\lambda_m s} \phi_m \langle x, \phi_m \rangle, \phi_n \right\rangle \phi_n \\ &= \sum_{n=1}^{\infty} e^{\lambda_n t} e^{\lambda_n s} \langle x, \phi_n \rangle \phi_n = T(t+s)x. \end{aligned}$$

Clearly, $T(0) = I$, and the strong continuity follows from the following calculation: For $t \leq 1$ we have

$$\begin{aligned}\|T(t)x - x\|^2 &= \sum_{n=1}^{\infty} |e^{\lambda_n t} - 1|^2 |\langle x, \phi_n\rangle|^2 \\ &= \sum_{n=1}^{N} |e^{\lambda_n t} - 1|^2 |\langle x, \phi_n\rangle|^2 + \sum_{n=N+1}^{\infty} |e^{\lambda_n t} - 1|^2 |\langle x, \phi_n\rangle|^2 \\ &\leq \sup_{1\leq n\leq N} |e^{\lambda_n t} - 1|^2 \sum_{n=1}^{N} |\langle x, \phi_n\rangle|^2 + K \sum_{n=N+1}^{\infty} |\langle x, \phi_n\rangle|^2\end{aligned}$$

for $K = \sup_{0\leq t\leq 1, n\geq 1} |e^{\lambda_n t} - 1|^2$. For any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |\langle x, \phi_n\rangle|^2 < \frac{\varepsilon}{2K}$$

and we can choose $t_0 \leq 1$ such that $\sup\limits_{1\leq n\leq N} |e^{\lambda_n t_0} - 1|^2 \leq \frac{\varepsilon}{2\|x\|^2}$. Thus for $t \in [0, t_0]$ we have

$$\|T(t)x - x\|^2 \leq \frac{\varepsilon}{2\|x\|^2} \sum_{n=1}^{N} |\langle x, \phi_n\rangle|^2 + K\frac{\varepsilon}{2K} \leq \varepsilon,$$

which shows that $(T(t))_{t\geq 0}$ is strongly continuous. Thus (5.7) defines a C_0-semigroup if and only if $\sup\limits_{n\geq 1} \operatorname{Re} \lambda_n < \infty$.

We show that Example 5.1.1 is a special case of the previous example.

Example 5.1.4 (Example 5.1.1 continued)**.** We begin with the remark that in Example 5.1.3 the enumeration started at $n = 1$, but that this is unimportant. We might as well as have started at $n = 0$.

We define the functions $\phi_0(\zeta) = 1, \phi_n(\zeta) = \sqrt{2}\cos(n\pi\zeta)$. It is well-known that $\{1, \sqrt{2}\cos(n\pi\cdot), n \geq 1\}$ is an orthonormal basis of $L^2(0,1)$. Hence every $x_0 \in L^2(0,1)$ can be written as

$$x_0 = \sum_{n=0}^{\infty} \langle x_0, \phi_n\rangle \phi_n.$$

Further, the sequence $\{\alpha_n\}$ of Example 5.1.1 satisfies $\alpha_0 = \int_0^1 x_0(\zeta)d\zeta = \langle x_0, \phi_0\rangle$ and

$$\alpha_n = 2\int_0^1 x_0(\zeta)\cos(n\pi\zeta)\, d\zeta = \sqrt{2}\langle x_0, \phi_n\rangle, \qquad n \geq 1.$$

Thus (5.6) can be written as

$$x(\cdot, t) = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t} \langle x_0, \phi_n\rangle \phi_n(\cdot).$$

A comparison with (5.7) shows that the mapping $x_0 \mapsto x(\cdot, t)$ given at the end of Example 5.1.1 defines a strongly continuous semigroup.

As mentioned before, any exponential of a matrix defines a strongly continuous semigroup. It turns out that semigroups share many properties with these exponentials.

Theorem 5.1.5. *A strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X has the following properties:*

1. $\|T(t)\|$ *is bounded on every finite sub-interval of* $[0, \infty)$*;*
2. *The mapping $t \mapsto T(t)$ is strongly continuous on the interval* $[0, \infty)$*;*
3. *For all $x \in X$ we have that $\frac{1}{t}\int_0^t T(s)xds \to x$ as $t \downarrow 0$;*
4. *If $\omega_0 = \inf_{t>0}(\frac{1}{t}\log\|T(t)\|)$, then $\omega_0 = \lim_{t\to\infty}(\frac{1}{t}\log\|T(t)\|) < \infty$;*
5. *For every $\omega > \omega_0$, there exists a constant M_ω such that for every $t \geq 0$ we have $\|T(t)\| \leq M_\omega e^{\omega t}$.*

The constant ω_0 is called the growth bound *of the semigroup.*

Proof. 1. First we show that $\|T(t)\|$ is bounded on some neighborhood of the origin, that is, there exist $\delta > 0$ and $M > 1$ depending on δ such that

$$\|T(t)\| \leq M \qquad \text{for } t \in [0, \delta].$$

If this does not hold, then there exists a sequence $\{t_n\}$, $t_n \downarrow 0$ such that $\|T(t_n)\| \geq n$. Hence, by the uniform boundedness principle, there exists an element $x \in X$ such that $\{\|T(t_n)x\|, n \in \mathbb{N}\}$ is unbounded; but this contradicts the strong continuity at the origin. If we set $t = m\delta + \tau$ with $0 < \tau \leq \delta$, then

$$\begin{aligned}\|T(t)\| &= \|T(m\delta)T(\tau)\| = \|T(\delta)^m T(\tau)\| \\ &\leq \|T(\delta)\|^m \|T(\tau)\| \leq M^{1+m} \leq M M^{t/\delta} = Me^{\omega t},\end{aligned} \tag{5.9}$$

where $\omega = \delta^{-1}\log M$.

2. For fixed $x \in X$, $t > 0$ and $s \geq 0$, using inequality (5.9) we have

$$\|T(t+s)x - T(t)x\| \leq \|T(t)\|\|T(s)x - x\| \leq Me^{\omega t}\|T(s)x - x\|.$$

Hence by the strong continuity of $T(\cdot)$ we may conclude that

$$\lim_{s\downarrow 0}\|T(t+s)x - T(t)x\| = 0.$$

Moreover, for $x \in X$, $t > 0$ and $\tau \geq 0$ sufficiently small, we have

$$\|T(t-\tau)x - T(t)x\| \leq \|T(t-\tau)\|\|x - T(\tau)x\|.$$

Thus $\lim_{s\uparrow 0}\|T(t+s)x - T(t)x\| = 0$, and the mapping $t \mapsto T(t)x$ is continuous on the interval $[0,\infty)$.

3. Let $x \in X$ and $\varepsilon > 0$. By the strong continuity of $(T(t))_{t\geq 0}$ we can choose $\tau > 0$ such that $\|T(s)x - x\| \leq \varepsilon$ for all $s \in [0,\tau]$. For $t \in [0,\tau]$ we have that

$$\begin{aligned}\|\frac{1}{t}\int_0^t T(s)x\,ds - x\| &= \|\frac{1}{t}\int_0^t [T(s)x - x]ds\| \\ &\leq \frac{1}{t}\int_0^t \|T(s)x - x\|ds \leq \frac{1}{t}\int_0^t \varepsilon ds = \varepsilon.\end{aligned}$$

4. Let $t_0 > 0$ be a fixed number and $M = \sup_{t\in[0,t_0]} \|T(t)\|$. For every $t \geq t_0$ there exists $n \in \mathbb{N}$ such that $nt_0 \leq t < (n+1)t_0$. Consequently,

$$\begin{aligned}\frac{\log\|T(t)\|}{t} &= \frac{\log\|T(nt_0)T(t-nt_0)\|}{t} = \frac{\log\|T^n(t_0)T(t-nt_0)\|}{t} \\ &\leq \frac{n\log\|T(t_0)\|}{t} + \frac{\log M}{t} \\ &= \frac{\log\|T(t_0)\|}{t_0}\cdot\frac{nt_0}{t} + \frac{\log M}{t}.\end{aligned}$$

The latter term is less than or equal to $\frac{\log\|T(t_0)\|}{t_0} + \frac{\log M}{t}$ if $\log\|T(t_0)\|$ is positive, and it is less than or equal to $\frac{\log\|T(t_0)\|}{t_0}\frac{t-t_0}{t} + \frac{\log M}{t}$ if $\log\|T(t_0)\|$ is negative. Thus

$$\limsup_{t\to\infty}\frac{\log\|T(t)\|}{t} \leq \frac{\log\|T(t_0)\|}{t_0} < \infty,$$

and since t_0 is arbitrary, we have that

$$\limsup_{t\to\infty}\frac{\log\|T(t)\|}{t} \leq \inf_{t>0}\frac{\log\|T(t)\|}{t} \leq \liminf_{t\to\infty}\frac{\log\|T(t)\|}{t}.$$

Thus

$$\omega_0 = \inf_{t>0}\frac{\log\|T(t)\|}{t} = \lim_{t\to\infty}\frac{\log\|T(t)\|}{t} < \infty.$$

5. If $\omega > \omega_0$, then there exists a $t_0 > 0$ such that

$$\frac{\log\|T(t)\|}{t} < \omega \qquad \text{for } \ t \geq t_0;$$

that is, $\|T(t)\| \leq e^{\omega t}$ for $t \geq t_0$. Moreover, there exists a constant $M_0 \geq 1$ such that $\|T(t)\| \leq M_0$ for $0 \leq t \leq t_0$. Defining

$$M_\omega = \begin{cases} M_0 & \omega \geq 0, \\ e^{-\omega t_0}M_0 & \omega < 0, \end{cases}$$

we obtain the stated result. □

5.2 Infinitesimal generators

If A is an $n \times n$-matrix, then the semigroup $\left(e^{At}\right)_{t\geq 0}$ is directly linked to A via

$$A = \left.\left(\frac{d}{dt}e^{At}\right)\right|_{t=0}.$$

Next we associate in a similar way an operator A to a C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 5.2.1. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. If the following limit exists

$$\lim_{t\downarrow 0}\frac{T(t)x_0 - x_0}{t}, \tag{5.10}$$

then we say that x_0 is an element of the *domain* of A, shortly $x_0 \in D(A)$, and we define Ax_0 as

$$Ax_0 = \lim_{t\downarrow 0}\frac{T(t)x_0 - x_0}{t}. \tag{5.11}$$

We call A the *infinitesimal generator* of the strongly continuous semigroup $(T(t))_{t\geq 0}$.

We prove next that for every $x_0 \in D(A)$ the function $t \mapsto T(t)x_0$ is differentiable. This will enable us to link a strongly continuous semigroup (uniquely) to an abstract differential equation.

Theorem 5.2.2. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Hilbert space X with infinitesimal generator A. Then the following results hold:*

1. *For $x_0 \in D(A)$ and $t \geq 0$ we have $T(t)x_0 \in D(A)$;*
2. $\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0$ *for* $x_0 \in D(A)$, $t \geq 0$;
3. $\frac{d^n}{dt^n}(T(t)x_0) = A^nT(t)x_0 = T(t)A^nx_0$ *for* $x_0 \in D(A^n)$, $t \geq 0$;
4. $T(t)x_0 - x_0 = \int_0^t T(s)Ax_0ds$ *for* $x_0 \in D(A)$;
5. $\int_0^t T(s)xds \in D(A)$ *and* $A\int_0^t T(s)xds = T(t)x - x$ *for all* $x \in X$, *and* $D(A)$ *is dense in* X;
6. *A is a closed linear operator.*

Proof. 1, 2, 3. First we prove part 1 and part 2. Let $s > 0$ and consider

$$\frac{T(t+s)x_0 - T(t)x_0}{s} = T(t)\frac{(T(s)-I)x_0}{s} = \frac{T(s)-I}{s}T(t)x_0.$$

If $x_0 \in D(A)$, the limit of the second term exists as $s \downarrow 0$, and hence the other limits exist as well. In particular, $T(t)x_0 \in D(A)$ and the strong right derivative of $T(t)x_0$ equals $AT(t)x_0 = T(t)Ax_0$.

For $t > 0$ and $s > 0$ sufficiently small, we have

$$\frac{T(t-s)x_0 - T(t)x_0}{-s} = T(t-s)\frac{(T(s)-I)x_0}{s}.$$

Hence the strong left derivative exists and equals $T(t)Ax_0$. Part 3 follows by induction.

4. Let $x_0 \in D(A)$. By part 2 we have that the derivative of $T(t)x_0$ equals $T(t)Ax_0$. Thus

$$\int_0^t T(s)Ax_0 ds = \int_0^t \frac{dT(s)x_0}{ds} ds = T(t)x_0 - x_0,$$

which proves the assertion.

5. For every $x \in X$ we have

$$\frac{T(s)-I}{s}\int_0^t T(u)xdu = \frac{1}{s}\int_0^t T(s+u)xdu - \frac{1}{s}\int_0^t T(u)xdu.$$

Substituting $\rho = s + u$ in the second integral, we obtain

$$\begin{aligned}
\frac{T(s)-I}{s}\int_0^t T(u)xdu &= \frac{1}{s}\int_s^{t+s} T(\rho)xd\rho - \frac{1}{s}\int_0^t T(u)xdu \\
&= \frac{1}{s}\left(\int_t^{t+s} T(\rho)xd\rho + \int_s^t T(\rho)xd\rho - \int_s^t T(u)xdu - \int_0^s T(u)xdu\right) \\
&= \frac{1}{s}\left(\int_0^s (T(t+u) - T(u))xdu\right) \\
&= \frac{1}{s}\int_0^s T(u)(T(t)-I)xdu.
\end{aligned}$$

Now, as $s \downarrow 0$, the right-hand side tends to $(T(t) - I)x$ (see Theorem 5.1.5.3). Hence

$$\int_0^t T(u)xdu \in D(A) \quad \text{and} \quad A\int_0^t T(u)xdu = (T(t)-I)x.$$

Furthermore, by Theorem 5.1.5.3 we have $\frac{1}{t}\int_0^t T(u)xdu \to x$ as $t \downarrow 0$. Hence for any $x \in X$, there exists a sequence $\{x_n\}$ in $D(A)$ such that $x_n \to x$. This shows that $\overline{D(A)} = X$.

6. In order to prove that A is closed, we choose $\{x_n\}_{n\in\mathbb{N}}$ as a sequence in $D(A)$ converging to x such that $\{Ax_n\}$ converges to y. Then $\|T(s)Ax_n - T(s)y\| \leq Me^{\omega s}\|Ax_n - y\|$ and thus $T(s)Ax_n \to T(s)y$ uniformly on $[0,t]$. Now, as $x_n \in D(A)$, we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds.$$

Thus we obtain

$$T(t)x - x = \int_0^t T(s)yds,$$

and therefore

$$\lim_{t\downarrow 0}\frac{T(t)x-x}{t}=\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(s)yds=y.$$

Hence $x\in D(A)$ and $Ax=y$, which proves that A is closed. □

Theorem 5.2.2 implies in particular that for every $x_0\in D(A)$ the function x defined by $x(t)=T(t)x_0$ satisfies the abstract differential equation $\dot{x}(t)=Ax(t)$.

Definition 5.2.1 implies that every strongly continuous semigroup has a unique generator. It is not hard to show that every generator belongs to a unique semigroup.

Theorem 5.2.3. *Let $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ be C_0-semigroups with generators A_1 and A_2, respectively. If $A_1=A_2$, then $T_1(t)=T_2(t)$ for every $t\geq 0$.*

Proof. Let $x_0\in D(A_1)=D(A_2)$ and consider for a fixed $t>0$ the continuous function $f(s)=T_1(t-s)T_2(s)x_0$, for $s\in[0,t]$. By Theorem 5.2.2 we have that $T_2(s)x_0\in D(A_1)$. Thus we may differentiate f with respect to $s\in(0,t)$. Using Theorem 5.2.2.2, we find

$$\frac{df(s)}{ds}=-T_1(t-s)A_1T_2(s)x_0+T(t-s)A_2T_2(s)x_0=0,$$

since $A_1=A_2$. Thus the continuous function f is constant on $[0,t]$, which implies

$$T_1(t)x_0=f(0)=f(t)=T_2(t)x_0.$$

This shows that the semigroups are equal on the dense set $D(A_1)$. The boundedness of the operators implies that they are equal for all $x_0\in X$. □

The infinitesimal generator of a C_0-semigroup is always a closed operator. The following proposition shows that every s whose real part is larger than the growth bound lies in the resolvent set of this generator. In particular, this implies that the *resolvent set*, denoted by $\rho(A)$, is non-empty. If $sI-A$ is invertible, then its inverse, $(sI-A)^{-1}$, is called the *resolvent operator*. The following proposition also shows that the resolvent is just the Laplace transform of the semigroup.

Proposition 5.2.4. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with infinitesimal generator A and with growth bound ω_0, $\omega>\omega_0$ and $\|T(t)\|\leq Me^{\omega t}$. If $Re(s)>\omega$, then $s\in\rho(A)$, and for all $x\in X$ the following results hold:*

1. $(sI-A)^{-1}x=\int_0^\infty e^{-st}T(t)x\,dt$ *and* $\|(sI-A)^{-1}\|\leq\dfrac{M}{\alpha-\omega}$; $\alpha=\mathrm{Re}(s)$;
2. *For $\lambda,\mu\in\rho(A)$ we have the* resolvent identity

$$(\lambda-\mu)(\lambda I-A)^{-1}(\mu I-A)^{-1}=(\mu I-A)^{-1}-(\lambda I-A)^{-1}; \tag{5.12}$$

3. *The mapping $s\mapsto(sI-A)^{-1}$ is analytic on $\rho(A)$.*

Proof. 1. Let

$$R_s x := \int_0^\infty e^{-st} T(t) x dt, \qquad x \in X, \ \mathrm{Re}(s) > \omega. \tag{5.13}$$

This operator is well defined, since by Theorem 5.1.5

$$\|e^{-st} T(t) x\| \le M e^{(\omega-\alpha)t} \|x\|, \qquad \text{where } \ \alpha = \mathrm{Re}(s)$$

and $T(\cdot)x$ is a continuous function. Combining this estimate with (5.13) we find

$$\|R_s x\| \le M \int_0^\infty e^{-(\alpha-\omega)t} \|x\| \, dt = \frac{M}{\alpha - \omega} \|x\|, \tag{5.14}$$

and therefore R_s is bounded. It remains to show that R_s equals the resolvent operator. For $x \in X$ the following equalities hold:

$$\begin{aligned}
&\frac{T(\tau) - I}{\tau} R_s x = \frac{1}{\tau} \int_0^\infty e^{-st} [T(\tau + t) - T(t)] x \, dt \\
&= \frac{1}{\tau} \left(\int_\tau^\infty e^{-s(u-\tau)} T(u) x \, du - \int_0^\infty e^{-st} T(t) x \, dt \right) \\
&= \frac{1}{\tau} \left(e^{s\tau} \int_0^\infty e^{-su} T(u) x \, du - e^{s\tau} \int_0^\tau e^{-su} T(u) x \, du - \int_0^\infty e^{-st} T(t) x \, dt \right) \\
&= \frac{e^{s\tau} - 1}{\tau} \int_0^\infty e^{-st} T(t) x \, dt - \frac{e^{s\tau}}{\tau} \int_0^\tau e^{-st} T(t) x \, dt.
\end{aligned}$$

Since $(e^{-st} T(t))_{t \ge 0}$ is a strongly continuous semigroup (see Exercise 5.3), we may use Theorem 5.1.5.3 to conclude that the second term on the right-hand side converges to x for $\tau \downarrow 0$. Thus for all $x \in X$,

$$A R_s x = \lim_{\tau \downarrow 0} \left(\frac{T(\tau) - I}{\tau} \right) R_s x = s R_s x - x. \tag{5.15}$$

Furthermore, by Theorem 5.2.2 we find for every $x \in D(A)$ that

$$R_s A x = \int_0^\infty e^{-st} T(t) A x \, dt = \left[e^{-st} T(t) x \right]_{t=0}^\infty - (-s) \int_0^\infty e^{-st} T(t) x \, dt = -x + s R_s x,$$

where we used integration by parts. Thus we find that $R_s(sI - A)x = x$ for all $x \in D(A)$. Further, equation (5.15) shows $(sI - A) R_s x = x$ for all $x \in X$ and thus

$$(sI - A)^{-1} = R_s.$$

2. This follows easily from the equality

$$\begin{aligned}
(\lambda - \mu)(\lambda I - A)^{-1}(\mu I - A)^{-1} &= (\lambda I - A)^{-1}(\lambda I - A + A - \mu I)(\mu I - A)^{-1} \\
&= (\mu I - A)^{-1} - (\lambda I - A)^{-1}.
\end{aligned}$$

Part 3 is proved in Example A.2.2. □

5.3 Abstract differential equations

Theorem 5.2.2.2 shows that for $x_0 \in D(A)$ the function $x(t) = T(t)x_0$ is a solution of the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{5.16}$$

Definition 5.3.1. A differentiable function $x : [0, \infty) \to X$ is called a *classical solution* of (5.16) if for all $t \geq 0$ we have $x(t) \in D(A)$ and equation (5.16) is satisfied.

Using Theorem 5.2.2 it is not hard to show that the classical solution is uniquely determined for $x_0 \in D(A)$.

Lemma 5.3.2. *Let A be the infinitesimal generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$. Then for every $x_0 \in D(A)$, the map $t \mapsto T(t)x_0$ is the unique classical solution of* (5.16).

Proof. Let x be a classical solution of (5.16), and define for $s \in [0, t]$ the function $z(s) = T(t-s)x(s)$. Then for $s \in (0, t)$ and $h \in \mathbb{R}$ such that $s + h \in (0, t)$, we have that

$$\frac{z(s+h) - z(s)}{h} = \frac{T(t-s-h)x(s) - T(t-s)x(s)}{h} + T(t-s-h)\frac{x(s+h) - x(s)}{h}.$$

We want to show that the limit exists for $h \to 0$. The limit of the first term on the right-hand side exists, since $x(s) \in D(A)$, see Theorem 5.2.2.2. The limit of the second term on the right-hand side exists, since x is differentiable and since the semigroup is (strongly) continuous. Thus z is differentiable, and

$$\dot{z}(s) = -AT(t-s)x(s) + T(t-s)Ax(s) = 0, \qquad s \in (0, t).$$

In other words, z is constant on $(0, t)$, and since it is continuous we find

$$x(t) = z(t) = z(0) = T(t)x(0) = T(t)x_0,$$

which proves the assertion. □

Definition 5.3.3. A continuous function $x : [0, \infty) \to X$ is called a *mild solution* of (5.16) if $\int_0^t x(s)\, ds \in D(A)$, $x(0) = x_0$ and

$$x(t) - x(0) = A \int_0^t x(\tau)\, d\tau, \qquad \text{for all } t \geq 0. \tag{5.17}$$

Using Theorem 5.2.2, it is easy to see that $t \mapsto T(t)x_0$ is a mild solution of (5.16) for every $x_0 \in X$. Further, in Exercise 5.6 we show that the mild solution of (5.16) is uniquely determined.

Finally, we return to the p.d.e. of Examples 5.1.1 and 5.1.4. We constructed a C_0-semigroup and showed that the semigroup solves an abstract differential equation. A natural question is how this abstract differential equation is related to the p.d.e. (5.2). The mild solution x of (5.16) takes at every time t values in a Hilbert space X. For the p.d.e. (5.2) we chose $X = L^2(0,1)$, see Example 5.1.1. Thus $x(t)$ is a function of $\zeta \in [0,1]$. Writing the abstract differential equation down using two variables, we obtain

$$\frac{\partial x}{\partial t}(\zeta, t) = Ax(t, \zeta).$$

Comparing this with (5.2), A must equal $\frac{d^2}{d\zeta^2}$. Since for $x_0 \in D(A)$ the mild solution is a classical solution, the boundary conditions must be a part of the domain of A. So the operator A associated to the p.d.e. (5.2) is given by

$$\begin{aligned} Ah &= \frac{d^2 h}{d\zeta^2} \quad \text{with} \\ D(A) &= \Big\{ h \in L^2(0,1) \mid h, \tfrac{dh}{d\zeta} \text{ are absolutely continuous,} \\ &\qquad \tfrac{d^2 h}{d\zeta^2} \in L^2(0,1) \ \text{ and } \tfrac{dh}{d\zeta}(0) = 0 = \tfrac{dh}{d\zeta}(1) \Big\}. \end{aligned} \tag{5.18}$$

In the next chapter we show that the operator A generates a C_0-semigroup and that this semigroup equals the one found in Example 5.1.4.

5.4 Exercises

5.1. Let X be a Hilbert space and $A \in \mathcal{L}(X)$. Show that

$$e^{At} := \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

defines a strongly continuous semigroup. Calculate the infinitesimal generator of this C_0-semigroup.

5.2. Show that in Theorem 5.1.5.5 the inequality $\|T(t)\| \le M_\omega e^{\omega t}$ does not have to hold for $\omega = \omega_0$.

Hint: Consider the C_0-semigroup $(e^{At})_{t\ge 0}$ with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

5.3. Suppose that $(T(t))_{t\ge 0}$ is a C_0-semigroup on the Hilbert space X.

a) Let $\lambda \in \mathbb{C}$, and show that $\left(e^{\lambda t} T(t)\right)_{t\ge 0}$ is also a C_0-semigroup. What is the growth bound of this semigroup?

b) Prove that the infinitesimal generator of $\left(e^{\lambda t} T(t)\right)_{t\ge 0}$ is $\lambda I + A$, where A is the infinitesimal generator of $(T(t))_{t\ge 0}$.

5.4. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X_1, with infinitesimal generator A and suppose that $\|T(t)\| \leq Me^{\omega t}$. Let $S \in \mathcal{L}(X_1, X_2)$ be invertible, where X_2 is another Hilbert space.

a) Define $T_2(t) = ST(t)S^{-1}$ and show that this is a C_0-semigroup on X_2.

b) Show that the infinitesimal generator of $(T_2(t))_{t\geq 0}$ is given by $A_2 = SAS^{-1}$ with $D(A_2) = \{z \in X_2 \mid S^{-1}z \text{ is an element of } D(A)\}$.

c) Show that $\|T_2(t)\| \leq M_2 e^{\omega t}$ for all $t \geq 0$.

5.5. Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Hilbert space X. Assume further that its growth bound ω_0 is negative. Let M be defined as

$$M = \int_0^\infty \|T(t)\| \, dt.$$

(a) Show that M is finite.

(b) Use Theorem 5.1.5 to show that $M^{-1} \leq -\omega_0$.

5.6. Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. Show that for every $x_0 \in X$, the map $t \mapsto T(t)x_0$ is the unique mild solution of (5.16).

5.5 Notes and references

The theory of strongly continuous semigroups started with the work of Hille, Phillips, and Yosida in the 1950's. By now it is a well-documented theory, and the results as formulated in this chapter can be found in many places. Good text book references are e.g. Curtain and Zwart [10], Engel and Nagel [15] and Pazy [44]. Moreover, the original sources Yosida [61] and Hille and Phillips [24] are still recommended.

Chapter 6

Contraction and Unitary Semigroups

In the previous chapter we introduced C_0-semigroups and their generators. We showed that every C_0-semigroup possesses an infinitesimal generator. In this section we study the other implication, i.e., when is a closed densely defined operator an infinitesimal generator of a C_0-semigroup? We remark that if A is bounded, then its exponential

$$e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \tag{6.1}$$

is the C_0-semigroup with A as its infinitesimal generator, see Exercise 5.1. In this chapter we restrict ourselves to infinitesimal generators of contraction semigroups on (separable) Hilbert spaces.

6.1 Contraction semigroups

Every C_0-semigroup satisfies $\|T(t)\| \leq Me^{\omega t}$ for some M and ω, see Theorem 5.1.5. In Example 5.1.3, the constant M equals 1, see (5.8), and thus the C_0-semigroup $(e^{-\omega t}T(t))_{t\geq 0}$ satisfies $\|e^{-\omega t}T(t)\| \leq 1$ for all $t \geq 0$. Semigroups with this special property are called contraction semigroups.

Definition 6.1.1. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X. Then $(T(t))_{t\geq 0}$ is called a *contraction semigroup*, if $\|T(t)\| \leq 1$ for every $t \geq 0$.

In Proposition 5.2.4 we showed that the resolvent of any generator is bounded by $M/\mathrm{Re}(s)$ for the real part of s sufficiently large. Operators with this property for $s \in (0, \infty)$ and $M = 1$ are studied next.

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*,
Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_6,

Lemma 6.1.2. *Let $A : D(A) \to X$ be a closed linear operator with dense domain such that $(0, \infty) \subset \rho(A)$ and*

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha}, \qquad \alpha > 0. \tag{6.2}$$

Then

$$\lim_{\alpha \to \infty} \alpha(\alpha I - A)^{-1} x = x \tag{6.3}$$

for all $x \in X$, where α is constrained to be real.

Proof. Let $x \in X$. Since the domain of A is dense in X, we can find $z \in D(A)$ such that $\|x - z\| \leq \varepsilon$, for any given $\varepsilon > 0$. Choose $\alpha_0 \in (0, \infty)$ such that $\|(\alpha I - A)^{-1}\| \leq \dfrac{\varepsilon}{\|Az\|}$ for all real $\alpha > \alpha_0$. Calculating $\|\alpha(\alpha I - A)^{-1}x - x\|$ for $\alpha > \alpha_0$ gives

$$\begin{aligned}
&\|\alpha(\alpha I - A)^{-1}x - x\| \\
&= \left\|\alpha(\alpha I - A)^{-1}x - \alpha(\alpha I - A)^{-1}z + \alpha(\alpha I - A)^{-1}z - z + z - x\right\| \\
&\leq \|\alpha(\alpha I - A)^{-1}(x - z)\| + \|(\alpha I - A + A)(\alpha I - A)^{-1}z - z\| + \|z - x\| \\
&\leq \frac{\alpha}{\alpha}\|x - z\| + \|(\alpha I - A)^{-1}Az\| + \|z - x\| \\
&\leq 3\varepsilon,
\end{aligned}$$

where we used (6.2). The above holds for every $\varepsilon > 0$. Thus $\lim_{\alpha \to \infty} \alpha(\alpha I - A)^{-1}x = x$, where α is real. □

The following Hille-Yosida theorem on the characterization of infinitesimal generators is very important, since it provides us with a necessary and sufficient condition for A to be the infinitesimal generator of a contraction semigroup. Note however, that in applications often the equivalent formulations of Theorems 6.1.7 and 6.1.8 are used.

Theorem 6.1.3 (Hille-Yosida Theorem)**.** *A necessary and sufficient condition for a closed, densely defined, linear operator A on a Hilbert space X to be the infinitesimal generator of a contraction semigroup is that $(0, \infty) \subset \rho(A)$ and*

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha} \qquad \text{for all } \alpha > 0. \tag{6.4}$$

Proof. Necessity: Follows directly from Proposition 5.2.4, since $\omega_0 \leq 0$ and $M = 1$. *Sufficiency*: Set $A_\alpha = \alpha^2(\alpha I - A)^{-1} - \alpha I$, $\alpha > 0$. Then $A_\alpha \in \mathcal{L}(X)$ and the corresponding C_0-semigroup is given by

$$T^\alpha(t) = e^{A_\alpha t} = e^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha^2 t)^n}{n!} (\alpha I - A)^{-n},$$

see Exercises 5.1 and 5.3. We show next that the strong limit of $T^\alpha(t)$ exists as $\alpha \to \infty$, and it is the desired semigroup, $T(t)$. First we show that $\|A_\alpha x - Ax\| \to 0$ as $\alpha \to \infty$ for $x \in D(A)$. Lemma 6.1.2 implies

$$\alpha(\alpha I - A)^{-1}x \to x \qquad \text{as } \alpha \to \infty \text{ for all } x \in X.$$

Moreover, $A_\alpha x = \alpha(\alpha I - A)^{-1}Ax$, and thus $A_\alpha x \to Ax$ as $\alpha \to \infty$ for $x \in D(A)$. Note that

$$\|T^\alpha(t)\| \leq e^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha^2 t)^n}{n!} \frac{1}{\alpha^n} = 1. \tag{6.5}$$

Thus the semigroup $(T^\alpha(t))_{t\geq 0}$ is uniformly bounded on $[0, \infty)$. Now,

$$(\alpha I - A)^{-1}(\mu I - A)^{-1} = (\mu I - A)^{-1}(\alpha I - A)^{-1},$$

and hence $A_\alpha A_\mu = A_\mu A_\alpha$ and $A_\alpha T^\mu(t) = T^\mu(t)A_\alpha$. For $x \in D(A)$, we obtain

$$\begin{aligned} T^\alpha(t)x - T^\mu(t)x &= \int_0^t \frac{d}{ds}(T^\mu(t-s)T^\alpha(s)x)ds \\ &= \int_0^t T^\mu(t-s)(A_\alpha - A_\mu)T^\alpha(s)xds \\ &= \int_0^t T^\mu(t-s)T^\alpha(s)(A_\alpha - A_\mu)xds. \end{aligned}$$

Thus for α and μ greater than 0 it follows that

$$\|T^\alpha(t)x - T^\mu(t)x\| \leq \int_0^t \|(A_\alpha - A_\mu)x\|ds = t\|(A_\alpha - A_\mu)x\|.$$

However, $\|(A_\alpha - A_\mu)x\| \to 0$ as $\alpha, \mu \to \infty$, since $A_\alpha x \to Ax$ as $\alpha \to \infty$. Thus $T^\alpha(t)x$ is a Cauchy sequence and so it converges to $T(t)x$, say. Using the uniform boundedness of $T^\alpha(t)x$ and the fact that $D(A)$ is dense in x we may extend this convergence to every $x \in X$. Clearly, $T(t) \in \mathcal{L}(X)$. Again, using the uniform boundedness of $T^\alpha(t)$ and the equation above, we see that $T^\alpha(t)x$ converges to $T(t)x$ uniformly on compact time intervals. Furthermore, from (6.5) we conclude that

$$\|T(t)x\| \leq \liminf_{\alpha\to\infty} \|T^\alpha(t)x\| \leq \liminf_{\alpha\to\infty} \|x\| = \|x\|.$$

It remains to show that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with infinitesimal generator A. For all $x \in X$ we have

$$T(t+s)x = \lim_{\alpha\to\infty} T^\alpha(t+s)x = \lim_{\alpha\to\infty} T^\alpha(t)T^\alpha(s)x = T(t)T(s)x.$$

In addition, $T(0) = I$, and the strong continuity is a consequence of the uniform convergence on compact intervals. For $x \in D(A)$ we obtain

$$\|T^\alpha(t)A_\alpha x - T(t)Ax\| \leq \|T^\alpha(t)\|\|A_\alpha x - Ax\| + \|T^\alpha(t)Ax - T(t)Ax\|$$

and hence $T^\alpha(t)A_\alpha x$ converges to $T(t)Ax$ as $\alpha \to \infty$, uniformly on compact intervals for $x \in D(A)$. Thus we may apply the limit $\alpha \to \infty$ to

$$T^\alpha(t)x - x = \int_0^t T^\alpha(s)A_\alpha x\, ds$$

to obtain

$$T(t)x - x = \int_0^t T(s)Ax\, ds \qquad \text{for } x \in D(A).$$

Therefore the infinitesimal generator $\tilde{A}$ of $(T(t))_{t\geq 0}$ is an extension of A, since

$$\tilde{A}x = \lim_{t\to 0^+} \frac{T(t)x - x}{t} = Ax \qquad\qquad \text{for } x \in D(A).$$

If $\alpha > 0$, then

$$(\alpha I - A)D(A) = X,$$

which implies

$$(\alpha I - \tilde{A})D(\tilde{A}) = X.$$

But $AD(A) = \tilde{A}D(A)$, and hence

$$(\alpha I - \tilde{A})D(A) = (\alpha I - \tilde{A})D(\tilde{A}).$$

Since $\alpha I - \tilde{A}$ is invertible, see Proposition 5.2.4, we see that $D(A) = D(\tilde{A})$, and this completes the proof of the theorem. □

Next we give a characterization of generators of contraction semigroups that does not require the explicit knowledge of the resolvent. Therefore we introduce the notion of dissipativity.

Definition 6.1.4. A linear operator $A : D(A) \subset X \to X$ is called *dissipative*, if

$$\operatorname{Re}\langle Ax, x\rangle \leq 0, \qquad x \in D(A). \tag{6.6}$$

We remark that a dissipative operator is in general neither closed nor densely defined.

Proposition 6.1.5. *A linear operator $A : D(A) \subset X \to X$ is dissipative if and only if*

$$\|(\alpha I - A)x\| \geq \alpha\|x\| \ \text{ for } x \in D(A), \alpha > 0. \tag{6.7}$$

Proof. We first assume that the operator A is dissipative. Using the Cauchy-Schwarz inequality, for $x \in D(A)$ and $\alpha > 0$ we have

$$\begin{aligned}\|(\alpha I - A)x\|\,\|x\| &\geq \operatorname{Re}\langle(\alpha I - A)x, x\rangle \\ &= \alpha\|x\|^2 - \operatorname{Re}\langle Ax, x\rangle \geq \alpha\|x\|^2,\end{aligned}$$

where we have used (6.6). Hence we have proved (6.7). Conversely, for $x \in D(A)$ and $\alpha > 0$, we have

$$\|(\alpha I - A)x\|^2 = \alpha^2\|x\|^2 - 2\alpha \mathrm{Re}\langle Ax, x\rangle + \|Ax\|^2.$$

Using (6.7), for all $\alpha > 0$ we conclude that

$$-2\alpha \mathrm{Re}\langle Ax, x\rangle + \|Ax\|^2 = \|(\alpha I - A)x\|^2 - \alpha^2\|x\|^2 \geq 0. \tag{6.8}$$

Dividing by α and letting α go to infinity, we find that (6.6) must hold. Since $x \in D(A)$ was arbitrary, we conclude that A is dissipative. □

If we additionally assume that the operator A is closed, then dissipativity implies that the range of $\alpha I - A$ is closed for all $\alpha > 0$.

Lemma 6.1.6. *If A is a closed, dissipative operator, then the range of $\alpha I - A$ is closed for all $\alpha > 0$.*

Proof. Let $\{z_n, n \in \mathbb{N}\}$ be a sequence in the range of $\alpha I - A$ such that $z_n \to z \in X$ for $n \to \infty$. We write $z_n = (\alpha I - A)x_n$ with $x_n \in D(A)$. By the dissipativity, see equation (6.7) we have

$$\|z_n - z_m\| = \|(\alpha I - A)(x_n - x_m)\| \geq \alpha\|x_n - x_m\|.$$

Since $\{z_n, n \in \mathbb{N}\}$ converges, we see that $\{x_n, n \in \mathbb{N}\}$ is a Cauchy sequence, and therefore also converges. We call this limit x. So $z_n = (\alpha I - A)x_n \to z$ and $x_n \to x$. Using the fact that A and thus $\alpha I - A$ is closed, we have that $z = (\alpha I - A)x$. Thus the range of $\alpha I - A$ is closed. □

Next we show that the generators of contraction semigroups are precisely those dissipative operators for which $I - A$ is surjective.

Theorem 6.1.7 (Lumer-Phillips Theorem)**.** *Let A be a linear operator with domain $D(A)$ on a Hilbert space X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A is dissipative and ran$(I - A) = X$.*

Proof. We first assume that A generates a contraction semigroup. Then the Hille-Yosida Theorem implies that $(0, \infty) \subset \rho(A)$ and

$$\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha} \qquad \text{for all } \alpha > 0.$$

Thus in particular, ran$(I - A) = X$ and

$$\|(\alpha I - A)x\| \geq \alpha\|x\| \;\text{ for } x \in D(A), \alpha > 0.$$

Now Proposition 6.1.5 implies that A is dissipative.

Conversely, using the Hille-Yosida Theorem it is sufficient to show that A is densely defined and closed, $(0, \infty) \subset \rho(A)$ and A satisfies (6.4). We first show that A is densely defined.

Let z be orthogonal to $D(A)$. Since the range of $(I - A)$ equals X, there exists an $x \in D(A)$ such that $z = (I - A)x$. Since z is orthogonal to any element in $D(A)$, we have in particular that $\langle z, x\rangle = 0$. Thus

$$0 = \langle z, x\rangle = \langle (I - A)x, x\rangle = \|x\|^2 - \langle Ax, x\rangle.$$

Taking real parts, we find

$$0 = \|x\|^2 - \mathrm{Re}\langle Ax, x\rangle.$$

However, the dissipativity of A now implies $\|x\| = 0$ or equivalently $x = 0$. Thus $z = (I - A)x = 0$, and therefore the domain of A is dense in X.

Using Proposition 6.1.5 we have

$$\|(I - A)x\| \geq \|x\| \quad \text{for } x \in D(A), \tag{6.9}$$

which implies that $I - A$ is injective. Combining this with the surjectivity, $I - A$ has an (algebraic) inverse. Using (6.9) once more we see that $(I - A)^{-1}$ is a bounded linear operator. The inverse of a bounded operator is a closed operator. Thus $I - A$ is closed and therefore A is a closed operator.

It remains to show that for any $\alpha > 0$, $\alpha I - A$ is surjective. If so, we can repeat the above argument, and conclude from (6.7) that (6.4) holds, and thus A generates a contraction semigroup.

We first show that the range is dense. Let z be orthogonal to the range of $\alpha I - A$. Since the range of $I - A$ equals X, we can find an $x \in D(A)$ such that $z = (I - A)x$. In particular, we find

$$0 = \langle z, (\alpha I - A)x\rangle = \langle (I - A)x, (\alpha I - A)x\rangle = \alpha\|x\|^2 - \alpha\langle Ax, x\rangle - \langle x, Ax\rangle + \|Ax\|^2.$$

Taking real parts,

$$0 = \alpha\|x\|^2 - \alpha\mathrm{Re}\langle Ax, x\rangle - \mathrm{Re}\langle x, Ax\rangle + \|Ax\|^2.$$

By the dissipativity all terms on the right-hand side are positive. Thus $x = 0$, and $z = (I - A)x = 0$. In other words, the range of $\alpha I - A$ is dense in X. By Lemma 6.1.6 we have that the range is closed, and thus the range of $\alpha I - A$ equals X. □

The following theorem gives another simple characterization of generators of contraction semigroups.

Theorem 6.1.8. *Let A be a linear, densely defined and closed operator on a Hilbert space X. Then A is the infinitesimal generator of a contraction semigroup $(T(t))_{t\geq 0}$ on X if and only if A and A^* are dissipative.*

Proof. We first assume that A generates a contraction semigroup. By Theorem 6.1.7, A is dissipative. As A is densely defined, the operator A^* is well-defined. By the Hille-Yosida Theorem, we have that $\|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha}$. Since the norm of the adjoint equals the norm of the operator, and since the inverse of the adjoint equals the adjoint of the inverse, we find that

$$\|(\alpha I - A^*)^{-1}\| = \|(\alpha I - A)^{-1}\| \leq \frac{1}{\alpha}.$$

This implies that for all $x \in D(A^*)$ we have $\|(\alpha I - A^*)x\| \geq \alpha\|x\|^2$. Proposition 6.1.5 gives that A^* is dissipative.

Next assume that A and A^* are dissipative. By Lemma 6.1.6, for every $\alpha > 0$ the range of $\alpha I - A$ is closed. Assume that there exists an $\alpha_0 > 0$ and a non-zero z such that z is orthogonal to the range of $\alpha_0 I - A$. Then for all $x \in D(A)$, we have

$$0 = \langle z, (\alpha_0 I - A)x\rangle = \alpha_0\langle z, x\rangle - \langle z, Ax\rangle. \tag{6.10}$$

By the definition of A^*, this implies that $z \in D(A^*)$ and $A^*z = \alpha_0 z$. Since A^* is dissipative, we have that

$$0 \geq \mathrm{Re}\langle A^*z, z\rangle = \langle \alpha_0 z, z\rangle = \alpha_0\|z\|^2.$$

This is a contradiction to the positivity of α_0. Hence for all $\alpha > 0$ the range of $\alpha I - A$ equals X. By Theorem 6.1.7 we conclude that A generates a contraction semigroup. □

Remark 6.1.9. Instead of assuming that A^* is dissipative it is sufficient to assume that A^* has no eigenvalues on the positive real axis.

Next we apply the obtained results to the example of the heated bar, see Example 5.1.1.

Example 6.1.10. The abstract formulation of Example 5.1.1 leads to the following operator, see (5.18),

$$\begin{aligned} Ah &= \frac{d^2h}{d\zeta^2} \quad \text{with} \\ D(A) &= \Big\{h \in L^2(0,1) \mid h, \tfrac{dh}{d\zeta} \text{ are absolutely continuous,} \\ &\qquad \tfrac{d^2h}{d\zeta^2} \in L^2(0,1) \text{ and } \tfrac{dh}{d\zeta}(0) = 0 = \tfrac{dh}{d\zeta}(1)\Big\}, \end{aligned} \tag{6.11}$$

Next we show that A generates a contraction semigroup on $L^2(0,1)$. A is dissipative, as

$$\begin{aligned} \langle h, Ah\rangle + \langle Ah, h\rangle &= \int_0^1 h(\zeta)\overline{\frac{d^2h}{d\zeta^2}(\zeta)} + \frac{d^2h}{d\zeta^2}(\zeta)\overline{h(\zeta)}\,d\zeta \\ &= \left(h(\zeta)\overline{\frac{dh}{d\zeta}(\zeta)} + \frac{dh}{d\zeta}(\zeta)\overline{h(\zeta)}\right)\Big|_0^1 - 2\int_0^1 \frac{dh}{d\zeta}(\zeta)\overline{\frac{dh}{d\zeta}(\zeta)}\,d\zeta \\ &= 0 - 2\int_0^1 \|\frac{dh}{d\zeta}(\zeta)\|^2\,d\zeta \leq 0, \end{aligned} \tag{6.12}$$

where we have used the boundary conditions. It remains to show that the range of $I - A$ equals $L^2(0,1)$, i.e., for every $f \in L^2(0,1)$ we have to find an $h \in D(A)$ such that $(I - A)h = f$. Let $f \in L^2(0,1)$ and define

$$h(\zeta) = \alpha \cosh(\zeta) - \int_0^\zeta \sinh(\zeta - \tau) f(\tau)\, d\tau, \tag{6.13}$$

where

$$\alpha = \frac{1}{\sinh(1)} \int_0^1 \cosh(1 - \tau) f(\tau)\, d\tau. \tag{6.14}$$

It is easy to see that h is an element of $L^2(0,1)$ and absolutely continuous. Furthermore, its derivative is given by

$$\frac{dh}{d\zeta}(\zeta) = \alpha \sinh(\zeta) - \int_0^\zeta \cosh(\zeta - \tau) f(\tau)\, d\tau.$$

This function is also absolutely continuous and satisfies the boundary condition $\frac{dh}{d\zeta}(0) = 0$. Using (6.14) we also have that $\frac{dh}{d\zeta}(1) = 0$. Furthermore,

$$\frac{d^2 h}{d\zeta^2}(\zeta) = \alpha \cosh(\zeta) - f(\zeta) - \int_0^\zeta \sinh(\zeta - \tau) f(\tau)\, d\tau = -f(\zeta) + h(\zeta),$$

where we have used (6.13). Thus $h \in D(A)$ and $(I - A)h = f$. This proves that for every $f \in L^2(0,1)$ there exists an $h \in D(A)$ such that $(I - A)h = f$. Thus the theorem of Lumer-Phillips implies that A generates a contraction semigroup, which by Theorem 5.2.3 is uniquely determined. It remains to show that this semigroup equals the one found in Examples 5.1.1 and 5.1.4.

Let $(T_1(t))_{t \geq 0}$ denote the semigroup of Example 5.1.4, and $(T_2(t))_{t \geq 0}$ the semigroup generated by the operator A as given in (6.11).

Let $\phi_n \in L^2(0,1)$ be given as $\phi_n(\zeta) = \sqrt{2}\cos(n\pi\zeta)$, $n \in \mathbb{N}$ and $\phi_0(\zeta) = 1$. It is easy to see that $\phi_n \in D(A)$. Furthermore, $A\phi_n = -n^2\pi^2\phi_n$. Since $\phi_n \in D(A)$, the abstract differential equation

$$\dot{x}(t) = Ax(t) \qquad x(0) = \phi_n$$

has a unique classical solution, $T_2(t)\phi_n$, see Lemma 5.3.2. By inspection, we see that $x(t) = e^{-n^2\pi^2 t}\phi_n$ satisfies this differential equation. Thus $T_2(t)\phi_n = e^{-n^2\pi^2 t}\phi_n$. Since the semigroup is a linear operator we find that

$$T_2(t)\left(\sum_{n=0}^N \alpha_n \phi_n\right) = \sum_{n=0}^N \alpha_n e^{-n^2\pi^2 t}\phi_n.$$

Thus on span $\{\phi_n\}$, $T_2(t)$ equals $T_1(t)$. Since span $\{\phi_n\}$ lies densely in $L^2(0,1)$, we conclude that $T_2(t) = T_1(t)$ for $t \geq 0$.

6.2 Groups and unitary groups

By definition, semigroups are only defined on the positive time axis. However, the simplest example of a semigroup, i.e., the exponential of a matrix, is also defined for $t < 0$. If the semigroup $(T(t))_{t\geq 0}$ can be extended to all $t \in \mathbb{R}$, then we say that $(T(t))_{t\in\mathbb{R}}$ is a group. We present the formal definition next.

Definition 6.2.1. Let X be a Hilbert space. $(T(t))_{t\in\mathbb{R}}$ is a *strongly continuous group*, or C_0*-group*, if the following holds:

1. For all $t \in \mathbb{R}$, $T(t)$ is a bounded linear operator on X;
2. $T(0) = I$;
3. $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \in \mathbb{R}$.
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \to 0$.

It is easy to see that the exponential of a matrix is a group. As for semigroups, we can define the infinitesimal generator of a group, see Definition 5.2.1.

Definition 6.2.2. Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on the Hilbert space X. The infinitesimal generator is defined as

$$Ax_0 = \lim_{t\to 0} \frac{T(t)x_0 - x_0}{t} \tag{6.15}$$

for $x_0 \in D(A)$. Here $D(A)$ consists of all $x_0 \in X$ for which this limit exists.

The following theorem characterizes generators of C_0-groups.

Theorem 6.2.3. *A is the infinitesimal generator of a C_0-group if and only if A and $-A$ are infinitesimal generators of C_0-semigroups.*

More precisely, if $(T(t))_{t\in\mathbb{R}}$ is a C_0-group with generator A, then $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ defined by

$$T_+(t) = T(t), \quad t \geq 0, \qquad \text{and} \qquad T_-(t) = T(-t) \quad t \geq 0, \tag{6.16}$$

are C_0-semigroups with infinitesimal generators A and $-A$, respectively. Conversely, if A and $-A$ generate C_0-semigroups $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$, then A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$, which is given by

$$T(t) = \begin{cases} T_+(t), & t \geq 0, \\ T_-(-t), & t \leq 0. \end{cases} \tag{6.17}$$

Proof. Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group. The semigroups $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ are defined by (6.16). It is easy to see that these are strongly continuous semigroups on X. We denote their infinitesimal generators by A_+ and A_-, respectively.

It is easy to see that $D(A) \subset D(A_+)$, $D(A) \subset D(A_-)$ and

$$Ax_0 = A_+x_0, \qquad x_0 \in D(A). \tag{6.18}$$

Next we show that these three domains are the same. Let $x_0 \in D(A_+)$, and consider the following equation for $t > 0$:

$$\frac{T(-t)x_0 - x_0}{t} + A_+x_0 = T(-t)\left(\frac{x_0 - T(t)x_0}{t} + A_+x_0\right) + A_+x_0 - T(-t)A_+x_0. \tag{6.19}$$

Since $x_0 \in D(A_+)$ and $(T(t))_{t\in\mathbb{R}}$ is strongly continuous, the right-hand side converges to zero for $t \downarrow 0$, and thus also the left-hand side converges to zero for $t \downarrow 0$. This implies that $x_0 \in D(A_-)$ and also that (6.15) exists, which shows $D(A_+) \subset D(A_-)$ and $D(A_+) \subset D(A)$. Since we already had that $D(A) \subset D(A_+)$, we conclude that $D(A_+) = D(A)$. Similarly, we can show that $D(A_-) = D(A)$. Furthermore, from (6.19) we conclude that $A_-x_0 = -Ax_0$. Combining this with (6.18), we have that $A_+ = A$, and $A_- = -A$. Thus both A and $-A$ are infinitesimal generators of C_0-semigroups.

Next we prove the other implication. Denote the semigroups generated by A and $-A$ by $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$, respectively. For $x_0 \in D(A)$ consider the continuous function $f(t) = T_+(t)T_-(t)x_0$, $t \geq 0$. By Theorem 5.2.2 this function is differentiable, and

$$\frac{df}{dt}(t) = T_+(t)AT_-(t)x_0 + T_+(t)(-A)T_-(t)x_0 = 0.$$

Thus $f(t) = f(0) = x_0$. Since $T_+(t)T_-(t)$ is a bounded operator, and since the domain of A is dense we have that

$$T_+(t)T_-(t) = I, \qquad t \geq 0. \tag{6.20}$$

Similarly, we find that $T_-(t)T_+(t) = I$. Now we define $(T(t))_{t\in\mathbb{R}}$ by (6.17). It is easy to see that $(T(t))_{t\in\mathbb{R}}$ satisfies properties 1, 2, and 4 of Definition 6.2.1. So it remains to show that property 3 holds. We prove this for $\tau < 0$, $t > 0$, and $\tau < -t$. The other cases are shown similarly.

We have that

$$T(t+\tau) = T_-(-t-\tau) = T_+(t)T_-(t)T_-(-t-\tau) = T_+(t)T_-(-\tau) = T(t)T(\tau),$$

where we have used (6.20) and the semigroup property of $(T_-(t))_{t\geq 0}$. □

Definition 6.2.4. A strongly continuous group $(T(t))_{t\in\mathbb{R}}$ is called a *unitary group* if $\|T(t)x\| = \|x\|$ for every $x \in X$ and every $t \in \mathbb{R}$.

We close this section with a characterization of the infinitesimal generator of a unitary group.

Theorem 6.2.5. *Let A be a linear operator on a Hilbert space X. Then A is the infinitesimal generator of the unitary group $(T(t))_{t\in\mathbb{R}}$ on X if and only if A and $-A$ generate a contraction semigroup.*

Proof. Let $(T(t))_{t\in\mathbb{R}}$ be the unitary group, then it is easy to see that

$$\|T(t)x_0\| = \|x_0\|, \quad t > 0, \qquad \text{and} \qquad \|T(-t)x_0\| = \|x_0\|, \quad t > 0.$$

Thus by Theorem 6.2.3, A and $-A$ generate contraction semigroups.

Assume next that A and $-A$ generate a contraction semigroup, then by Theorem 6.2.3, A generates a group which is given by (6.17). Let $x_0 \in X$ and $t > 0$, then

$$\begin{aligned}\|x_0\| = \|T(t)T(-t)x_0\| &\leq \|T(-t)x_0\| \text{ since } (T(t))_{t\geq 0} \text{ is a contraction semigroup} \\ &\leq \|x_0\|,\end{aligned}$$

where we have used that $(T(-t))_{t\geq 0}$ is a contraction semigroup. From these inequalities we obtain that $\|T(-t)x_0\| = \|x_0\|$, $t \geq 0$. Similarly, we can show that $\|T(t)x_0\| = \|x_0\|$. Thus $(T(t))_{t\in\mathbb{R}}$ is a unitary group. □

Remark 6.2.6. Another useful characterization of a unitary group is the following: A is the infinitesimal generator of a unitary group if and only if $A = -A^*$, i.e., A is *skew-adjoint*.

6.3 Exercises

6.1. Let A_0 be a self-adjoint, non-negative operator on the Hilbert space X. Prove that $A := -A_0$ is the infinitesimal generator of a contraction semigroup on X.

6.2. Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. Prove that $(T(t))_{t\geq 0}$ is a contraction semigroup if and only if

$$\langle Ax, x\rangle + \langle x, Ax\rangle \leq 0$$

for every $x \in D(A)$.

6.3. Let X be the Hilbert space $L^2(0,1)$ and define for $t \geq 0$ and $f \in X$ the following operator:

$$(T(t)f)(\zeta) = \begin{cases} f(t+\zeta), & \zeta \in [0,1],\ t+\zeta \leq 1, \\ 0, & \zeta \in [0,1],\ t+\zeta > 1. \end{cases} \tag{6.21}$$

(a) Show that $(T(t))_{t\geq 0}$ is a contraction semigroup on X.

(b) Prove that the infinitesimal generator of this semigroup is given by

$$Af = \frac{df}{d\zeta}, \tag{6.22}$$

with domain

$$D(A)=\left\{f\in L^2(0,1) \mid f \text{ is absolutely cont., } \frac{df}{d\zeta}\in L^2(0,1) \text{ and } f(1)=0\right\}.$$

Hint: In order to calculate the domain of the generator A, note that $D(A) = \operatorname{ran}\left((I-A)^{-1}\right)$ and use Proposition 5.2.4.

6.4. Consider the following operator on $X = L^2(0,1)$:

$$Af = \frac{df}{d\zeta}, \tag{6.23}$$

with domain

$$D(A)=\left\{f\in L^2(0,1) \mid f \text{ is absolutely cont., } \frac{df}{d\zeta}\in L^2(0,1) \text{ and } f(1)=f(0)\right\}.$$

(a) Show that this operator generates a unitary group on X.

(b) Find the expression for this group.

6.5. In this exercise, we give some simple p.d.e.'s which do not generate a C_0-semigroup. As state space we choose again $X = L^2(0,1)$.

(a) Let A be defined as

$$Af = \frac{df}{d\zeta}, \tag{6.24}$$

with domain

$$D(A)=\left\{f\in L^2(0,1) \mid f \text{ is absolutely cont. and } \frac{df}{d\zeta}\in L^2(0,1)\right\}.$$

Show that A is not an infinitesimal generator on $L^2(0,1)$.
Hint: Show that $\alpha I - A$ is not injective for any $\alpha > 0$.

(b) Let A be defined as

$$Af = \frac{df}{d\zeta}, \tag{6.25}$$

with domain

$$D(A)=\left\{f\in L^2(0,1) \mid f \text{ is absolutely cont., } \frac{df}{d\zeta}\in L^2(0,1) \text{ and } f(0)=0\right\}.$$

Show that A is not an infinitesimal generator on $L^2(0,1)$.
Hint: Show that the norm of $(\alpha I - A)^{-1} f$ for $f(\zeta) = 1, \zeta \in [0,1]$ does not satisfy the norm bound of Proposition 5.2.4.

6.4 Notes and references

Contraction and unitary semigroups appear often when applying semigroup theory to partial differential equations, see for instance Chapter 7. This implies that a characterization as given in the Lumer-Phillips Theorem is widely used. This theorem dates 50 years back, see [39] and [46]. There are many good references concerning this theorem, such as Curtain and Zwart [10], Engel and Nagel [15], Pazy [44], Yosida [61], and Hille and Phillips [24].

Chapter 7

Homogeneous Port-Hamiltonian Systems

In the previous two chapters we have formulated partial differential equations as abstract first order differential equations. Furthermore, we described the solutions of these differential equations via a strongly continuous semigroup. These differential equations were only weakly connected to the norm of the underlying state space. However, in this chapter we consider a class of differential equations for which there is a very natural state space norm. This natural choice enables us to show that the corresponding semigroup is a contraction semigroup.

7.1 Port-Hamiltonian systems

We begin by introducing a standard example of the class of port-Hamiltonian systems.

Example 7.1.1. We consider the *vibrating string* as depicted in Figure 7.1, which we have studied in Example 1.1.4 as well. The string is fixed at the left-hand

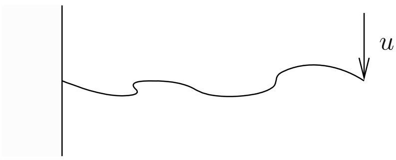

Figure 7.1: The vibrating string

side and may move freely at the right-hand side. We allow that a force u may be applied at the right-hand side. The model of the (undamped) vibrating string is

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*,
Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_7,

given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right), \tag{7.1}$$

where $\zeta \in [a,b]$ is the spatial variable, $w(\zeta,t)$ is the vertical position of the string at place ζ and time t, T is the Young's modulus of the string, and ρ is the mass density, which may vary along the string.

This system has the energy/Hamiltonian

$$E(t) = \frac{1}{2}\int_a^b \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta,t)\right)^2 + T(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)\right)^2 d\zeta. \tag{7.2}$$

Assuming that (7.1) possesses a (classical) solution, we may differentiate this Hamiltonian along the solutions of the partial differential equation

$$\begin{aligned}\frac{dE}{dt}(t) &= \int_a^b \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t)\frac{\partial^2 w}{\partial t^2}(\zeta,t) + T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\frac{\partial^2 w}{\partial \zeta \partial t}(\zeta,t)\, d\zeta \\ &= \int_a^b \frac{\partial w}{\partial t}(\zeta,t)\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right) + T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\frac{\partial^2 w}{\partial \zeta \partial t}(\zeta,t)\, d\zeta \\ &= \frac{\partial w}{\partial t}(b,t)T(b)\frac{\partial w}{\partial \zeta}(b,t) - \frac{\partial w}{\partial t}(a,t)T(a)\frac{\partial w}{\partial \zeta}(a,t), \end{aligned} \tag{7.3}$$

where we used integration by parts. Note that $\frac{\partial w}{\partial t}$ is the velocity, and $T\frac{\partial w}{\partial \zeta}$ is the force. Furthermore, velocity times force is *power* which by definition also equals the change of energy. Hence, (7.3) can be considered as a *power balance*, and the change of internal power only happens via the boundary of the spatial domain. The string is fixed at $\zeta = a$, that is, $\frac{\partial w}{\partial t}(a,t) = 0$. Furthermore, we have assumed that we control the force at $\zeta = b$. Thus the power balance (7.3) becomes

$$\frac{dE}{dt}(t) = \frac{\partial w}{\partial t}(b,t)u(t). \tag{7.4}$$

Next we apply a transformation to equation (7.1). We define $x_1 = \rho\frac{\partial w}{\partial t}$ (momentum) and $x_2 = \frac{\partial w}{\partial \zeta}$ (strain). Then (7.1) can equivalently be written as

$$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right) \\ &= P_1\frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right),\end{aligned} \tag{7.5}$$

where $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]$ and $\mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}$. The energy/Hamiltonian becomes in the new variables, see (7.2),

$$\begin{aligned} E(t) &= \frac{1}{2}\int_a^b \frac{x_1(\zeta,t)^2}{\rho(\zeta)} + T(\zeta)x_2(\zeta,t)^2 d\zeta \\ &= \frac{1}{2}\int_a^b \begin{bmatrix} x_1(\zeta,t) & x_2(\zeta,t) \end{bmatrix} \mathcal{H}(\zeta) \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} d\zeta. \end{aligned} \tag{7.6}$$

Various examples can be written in a similar form as the previous example, which is a particular example of a port-Hamiltonian system.

Definition 7.1.2. Let $P_1 \in \mathbb{K}^{n\times n}$ be invertible and self-adjoint, let $P_0 \in \mathbb{K}^{n\times n}$ be skew-adjoint, i.e., $P_0^* = -P_0$, and let $\mathcal{H} \in L^\infty([a,b];\mathbb{K}^{n\times n})$ such that $\mathcal{H}(\zeta)^* = \mathcal{H}(\zeta)$, $mI \le \mathcal{H}(\zeta) \le MI$ for a.e. $\zeta \in [a,b]$ and constants $m, M > 0$ independent of ζ. We equip the Hilbert space $X := L^2([a,b];\mathbb{K}^n)$ with the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^*\mathcal{H}(\zeta)f(\zeta)\,d\zeta. \tag{7.7}$$

Then the differential equation

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(\zeta,t)\right) + P_0\left(\mathcal{H}(\zeta)x(\zeta,t)\right). \tag{7.8}$$

is called a *linear, first order port-Hamiltonian system*. The associated *Hamiltonian* $E : [a,b] \to \mathbb{K}$ is given by

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{7.9}$$

Remark 7.1.3. We make the following observations concerning this definition.

1. Hamiltonian differential equations form an important subclass within ordinary and partial differential equations. They include linear and non-linear differential equations, and appear in many physical models. We restrict ourselves to linear differential equations of the type (7.8), and will normally omit the terms “linear, first order”.

2. In Hamiltonian differential equations, the associated Hamiltonian is normally denoted by H instead of E. Since in our examples the Hamiltonian will always be the energy, and since we want to have a clear distinction between $\mathcal{H}$ and the Hamiltonian, we have chosen for E.

3. Since the squared norm of X equals the energy associated to the linear, first order port-Hamiltonian system, we call X the *energy space*.

4. Since $mI \le \mathcal{H}(\zeta) \le MI$ for a.e. $\zeta \in [a,b]$, the standard L^2-norm is equivalent to the norm defined by (7.7), i.e., for all $f \in L^2([a,b];\mathbb{K}^n)$

$$\frac{m}{2}\|f\|_{L^2}^2 \le \|f\|_X^2 \le \frac{M}{2}\|f\|_{L^2}^2.$$

5. Real self-adjoint matrices are also called *symmetric*, and for complex self-adjoint the term *Hermitian* is used as well.

We have seen that the vibrating string is a port-Hamiltonian system. Next we show that the model of the Timoshenko beam is a port-Hamiltonian system as well.

Example 7.1.4. The model of the *Timoshenko beam* incorporates shear and rotational inertia effects in a vibrating beam. Its equations are given by

$$\begin{aligned} \rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left(K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right)\right), \quad \zeta\in(a,b),\ t\geq 0, \\ I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left(EI(\zeta)\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right)+K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right), \end{aligned} \tag{7.10}$$

where $w(\zeta,t)$ is the transverse displacement of the beam and $\phi(\zeta,t)$ is the rotation angle of a filament of the beam. The coefficients $\rho(\zeta)$, $I_\rho(\zeta)$, $EI(\zeta)$, and $K(\zeta)$ are the mass per unit length, the rotary moment of inertia of a cross section, the product of Young's modulus of elasticity and the moment of inertia of a cross section, and the shear modulus, respectively. The energy/Hamiltonian for this system is given by

$$\begin{aligned} E(t) = \frac{1}{2}\int_a^b \Bigg(& K(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right)^2+\rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta,t)\right)^2 \\ & +EI(\zeta)\left(\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right)^2+I_\rho(\zeta)\left(\frac{\partial \phi}{\partial t}(\zeta,t)\right)^2\Bigg)d\zeta. \end{aligned} \tag{7.11}$$

In order to show that the model of the Timoshenko beam is a port-Hamiltonian system, we introduce the following (physical) notation.

$$\begin{aligned} x_1(\zeta,t) &= \frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t) && \text{shear displacement} \\ x_2(\zeta,t) &= \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t) && \text{momentum} \\ x_3(\zeta,t) &= \frac{\partial \phi}{\partial \zeta}(\zeta,t) && \text{angular displacement} \\ x_4(\zeta,t) &= I_\rho(\zeta)\frac{\partial \phi}{\partial t}(\zeta,t) && \text{angular momentum.} \end{aligned}$$

Calculating the time derivative of the variables $x_1,\ldots,x_4$, we find by using (7.10),

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \zeta}\left(\frac{x_2(\zeta,t)}{\rho(\zeta)}\right)-\frac{x_4(\zeta,t)}{I_\rho(\zeta)} \\ \frac{\partial}{\partial \zeta}\left(K(\zeta)x_1(\zeta,t)\right) \\ \frac{\partial}{\partial \zeta}\left(\frac{x_4(\zeta,t)}{I_\rho(\zeta)}\right) \\ \frac{\partial}{\partial \zeta}\left(EI(\zeta)x_3(\zeta,t)\right)+K(\zeta)x_1(\zeta,t) \end{bmatrix} \tag{7.12}$$

Dropping for readability the coordinates ζ and t, we obtain

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right)$$
$$+ \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{7.13}$$

Formulating the energy/Hamiltonian in the variables $x_1, \dots, x_4$ is easier, see (7.11)

$$\begin{aligned} E(t) &= \frac{1}{2}\int_a^b K(\zeta)x_1(\zeta,t)^2 + \frac{1}{\rho(\zeta)}x_2(\zeta,t)^2 + EI(\zeta)x_3(\zeta,t)^2 + \frac{1}{I_\rho(\zeta)}x_4(\zeta,t)^2 d\zeta \\ &= \frac{1}{2}\int_a^b \left(\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{bmatrix}^* \begin{bmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{bmatrix}\right) d\zeta. \end{aligned} \tag{7.14}$$

This shows that the Timoshenko beam can be written as a port-Hamiltonian system on $X = L^2([a,b];\mathbb{R}^4)$. Next we calculate the power. Using the above notation and the model (7.10), we find that the power equals

$$\frac{dE}{dt}(t) = \left[K(\zeta)x_1(\zeta,t)\frac{x_2(\zeta,t)}{\rho(\zeta)} + EI(\zeta)x_3(\zeta,t)\frac{x_4(\zeta,t)}{I_\rho(\zeta)}\right]_a^b. \tag{7.15}$$

Again, the power goes via the boundary of the spatial domain.

As we have seen in the examples, the change of energy (power) of these systems was only possible via the boundary of its spatial domain. In the following theorem we show that this is a general property for any system which is of the form (7.8) with Hamiltonian (7.9).

Theorem 7.1.5. *Let x be a classical solution of the port-Hamiltonian system* (7.8) *with Hamiltonian* (7.9). *Then the following balance equation holds:*

$$\frac{dE}{dt}(t) = \frac{1}{2}\left[(\mathcal{H}(\zeta)x(\zeta,t))^* P_1 \mathcal{H}(\zeta)x(\zeta,t)\right]_a^b. \tag{7.16}$$

Proof. By using the partial differential equation, we obtain

$$\begin{aligned}\frac{dE}{dt}(t) &= \frac{1}{2}\int_a^b \frac{\partial x}{\partial t}(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta + \frac{1}{2}\int_a^b x(\zeta,t)^*\mathcal{H}(\zeta)\frac{\partial x}{\partial t}(\zeta,t)d\zeta \\ &= \frac{1}{2}\int_a^b \left(P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t)\right)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta \\ &\quad + \frac{1}{2}\int_a^b x(\zeta,t)^*\mathcal{H}(\zeta)\left(P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t)\right)d\zeta.\end{aligned}$$

Using now the fact that P_1 and $\mathcal{H}(\zeta)$ are self-adjoint, and P_0 is skew-adjoint, we write the last expression as

$$\begin{aligned}&\frac{1}{2}\int_a^b \left(\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t))\right)^* P_1\mathcal{H}(\zeta)x(\zeta,t) + (\mathcal{H}(\zeta)x(\zeta,t))^*\left(P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t))\right)d\zeta \\ &+ \frac{1}{2}\int_a^b -(\mathcal{H}(\zeta)x(\zeta,t))^* P_0\mathcal{H}(\zeta)x(\zeta,t) + (\mathcal{H}(\zeta)x(\zeta,t))^* P_0\mathcal{H}(\zeta)x(\zeta,t)d\zeta \\ &= \frac{1}{2}\int_a^b \frac{\partial}{\partial\zeta}\left[(\mathcal{H}(\zeta)x(\zeta,t))^* P_1\mathcal{H}(\zeta)x(\zeta,t)\right]d\zeta \\ &= \frac{1}{2}\left[(\mathcal{H}(\zeta)x(\zeta,t))^* P_1\mathcal{H}(\zeta)x(\zeta,t)\right]_a^b.\end{aligned}$$

Hence we have proved the theorem. □

The balance equation (7.16) is very important, and guides us in many problems. It explains the name *port-Hamiltonian system*. The system has a Hamiltonian (most times energy) and changes of this quantity can only occur via the boundary, i.e., the ports to the outside world. Note that (7.8) with $P_0 = 0$ is the infinite-dimensional counterpart of the finite-dimensional port-Hamiltonian system of Section 2.3. The J is replaced by $P_1\frac{\partial}{\partial\zeta}$, which is a skew-symmetric operator, and $\mathcal{H}$ is replaced by the operator, which multiplies by $\mathcal{H}(\cdot)$.

As we showed in the previous two chapters, a partial differential equation needs boundary conditions in order to possess a unique solution. In the next section we characterize those boundary conditions for which the port-Hamiltonian system generates a contraction semigroup.

7.2 Generation of contraction semigroups

In this section, we apply the general results presented in Chapters 5 and 6 to port-Hamiltonian systems, i.e., we consider partial differential equations of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)x(\zeta,t)\right) + P_0\mathcal{H}(\zeta)x(\zeta,t). \tag{7.17}$$

and we aim to characterize (homogeneous) boundary conditions such that (7.17) possesses a unique solution. In order to write (7.17) as an abstract differential equation, we "hide" the spatial dependence, and we write the p.d.e. as the (abstract) ordinary differential equation

$$\frac{dx}{dt}(t) = P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}x(t) \right) + P_0 \left(\mathcal{H}x(t) \right). \tag{7.18}$$

Hence we consider the operator

$$A_0 x := P_1 \frac{d}{d\zeta} \left(\mathcal{H}x \right) + P_0 \left(\mathcal{H}x \right) \tag{7.19}$$

on the state space

$$X = L^2([a,b]; \mathbb{K}^n) \tag{7.20}$$

with inner product

$$\langle f, g \rangle_X = \frac{1}{2} \int_a^b g(\zeta)^* \mathcal{H}(\zeta) f(\zeta)\, d\zeta, \tag{7.21}$$

and domain

$$D(A_0) = \left\{ x \in X \mid \mathcal{H}x \in H^1([a,b]; \mathbb{K}^n) \right\}. \tag{7.22}$$

Here $H^1([a,b]; \mathbb{K}^n)$ is the vector space of all functions from $[a,b]$ to $\mathbb{K}^n$, which are square integrable, absolutely continuous, and the derivative is again square integrable, that is,

$$\begin{aligned} &H^1([a,b]; \mathbb{K}^n) \\ &= \{ f \in L^2([a,b]; \mathbb{K}^n) \mid f \text{ is absolutely continuous and } \frac{df}{d\zeta} \in L^2([a,b]; \mathbb{K}^n) \}. \end{aligned}$$

Note, that A_0 as the maximal domain. In order to guarantee that (7.17) possesses a unique solution we have to add boundary conditions. It turns out that it is better to formulate them in the *boundary effort* and *boundary flow*, which are defined as

$$e_\partial = \frac{1}{\sqrt{2}} \left((\mathcal{H}x)(b) + (\mathcal{H}x)(a) \right) \quad \text{and} \quad f_\partial = \frac{1}{\sqrt{2}} \left(P_1 (\mathcal{H}x)(b) - P_1 (\mathcal{H}x)(a) \right), \tag{7.23}$$

respectively. Next, we show some properties of the operator A_0.

Lemma 7.2.1. *Consider the operator* A_0 *defined in* (7.19) *and* (7.22) *associated to a port-Hamiltonian system, that is, the assumptions of Definition* 7.1.2 *are satisfied. Then the following results hold:*

1. *Re* $\langle A_0 x, x \rangle_X = \frac{1}{4} (f_\partial^* e_\partial + e_\partial^* f_\partial)$.

2. *For every* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \mathbb{K}^{2n}$ *there exists an* $x_0 \in D(A_0)$ *such that* $\left[\begin{smallmatrix} (\mathcal{H}x_0)(b) \\ (\mathcal{H}x_0)(a) \end{smallmatrix}\right] = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$.

Proof. 1. For the differential operator A_0 and $x_0 \in D(A_0)$ we have

$$\langle A_0 x, x\rangle_X + \langle x, A_0 x\rangle_X = \frac{1}{2}\int_a^b x(\zeta)^* \mathcal{H}(\zeta)\left(P_1 \frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta) + P_0(\mathcal{H}x)(\zeta)\right) d\zeta + \frac{1}{2}\int_a^b \left(P_1 \frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta) + P_0(\mathcal{H}x)(\zeta)\right)^* \mathcal{H}(\zeta)x(\zeta)\, d\zeta.$$

Using the fact that P_1 is self-adjoint, and P_0 is skew-adjoint, we find

$$\begin{aligned}
&\langle A_0 x, x\rangle_X + \langle x, A_0 x\rangle_X \\
&= \frac{1}{2}\int_a^b (\mathcal{H}(\zeta)x(\zeta))^* \left(P_1 \frac{d}{d\zeta}(\mathcal{H}x)(\zeta)\right) + \left(\frac{d}{d\zeta}(\mathcal{H}x)(\zeta)\right)^* P_1 \mathcal{H}(\zeta)x(\zeta) d\zeta \\
&\quad + \frac{1}{2}\int_a^b (\mathcal{H}(\zeta)x(\zeta))^* (P_0\mathcal{H}(\zeta)x(\zeta)) - (\mathcal{H}(\zeta)x(\zeta))^* P_0 \mathcal{H}(\zeta)x(\zeta) d\zeta \\
&= \frac{1}{2}\int_a^b \frac{d}{d\zeta}\left((\mathcal{H}x)^*(\zeta) P_1 (\mathcal{H}x)(\zeta)\right) d\zeta \\
&= \frac{1}{2}\left((\mathcal{H}x)^*(b) P_1 (\mathcal{H}x)(b) - (\mathcal{H}x)^*(a) P_1 (\mathcal{H}x)(a)\right).
\end{aligned}$$

Combining this equality with (7.23), we obtain

$$\langle A_0 x, x\rangle_X + \langle x, A_0 x\rangle_X = \frac{1}{2}\left(f_\partial^* e_\partial + e_\partial^* f_\partial\right). \tag{7.24}$$

2. It is easy to see that x_0 defined as

$$x_0(\zeta) = \mathcal{H}^{-1}(\zeta)\left(y + \frac{\zeta - a}{b - a} u\right)$$

is an element of the domain of A_0 and satisfies the boundary conditions. □

The balance equation (7.16) shows that the boundary flow is not determined by x, but by $\mathcal{H}x$. Therefore, we formulate the boundary conditions in this variable. So we consider the boundary conditions

$$\tilde{W}_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} = 0, \qquad t \geq 0. \tag{7.25}$$

It turns out that formulating the boundary conditions directly in x or $\mathcal{H}x$ at $\zeta = a$ and $\zeta = b$ is not the best choice for characterizing generators of contraction semigroups. It is better to formulate them in the *boundary effort* and *boundary flow*, which are defined by (7.23). We write this as a matrix vector product, i.e.,

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_0 \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix}, \tag{7.26}$$

with $R_0 \in \mathbb{K}^{2n\times 2n}$ defined as

$$R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}. \tag{7.27}$$

Next, we show some properties of this transformation.

Lemma 7.2.2. *Let P_1 be self-adjoint and invertible in $\mathbb{K}^{n\times n}$, then the matrix $R_0 \in \mathbb{K}^{2n\times 2n}$ defined by (7.27) is invertible, and satisfies*

$$\begin{bmatrix} P_1 & 0 \\ 0 & -P_1 \end{bmatrix} = R_0^* \Sigma R_0, \tag{7.28}$$

where

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{7.29}$$

All possible matrices R which satisfy (7.28) are given by the formula $R = UR_0$, with U satisfying $U^*\Sigma U = \Sigma$.

Proof. We have that

$$\frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & I \\ -P_1 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} P_1 & 0 \\ 0 & -P_1 \end{bmatrix}.$$

Thus using the fact that P_1 is self-adjoint, we obtain that $R_0 := \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]$ satisfies (7.28). Since P_1 is invertible, the invertibility of R_0 follows from equation (7.28).

Let R be another solution of (7.28). Hence

$$R^*\Sigma R = \begin{bmatrix} P_1 & 0 \\ 0 & -P_1 \end{bmatrix} = R_0^* \Sigma R_0.$$

This can be written in the equivalent form

$$R_0^{-*} R^* \Sigma R R_0^{-1} = \Sigma.$$

Calling $RR_0^{-1} = U$, we obtain $U^*\Sigma U = \Sigma$ and $R = UR_0$, which proves the assertion. □

Since the matrix R_0 is invertible, we can write any condition which is formulated in $(\mathcal{H}x)(b)$ and $(\mathcal{H}x)(a)$ into an equivalent condition which is formulated in f_∂ and e_∂.

Using (7.26), we write the boundary condition (7.25) (equivalently) as

$$W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0, \tag{7.30}$$

where $W_B = \tilde{W}_B R_0^{-1}$. Thus we study the operator

$$Ax := P_1 \frac{d}{d\zeta} (\mathcal{H}x) + P_0 (\mathcal{H}x) \tag{7.31}$$

with domain

$$D(A) = \{x \in L^2([a,b];\mathbb{K}^n) \mid \mathcal{H}x \in H^1([a,b];\mathbb{K}^n),\ W_B \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] = 0\}. \tag{7.32}$$

In Theorem 7.2.4 we characterize the boundary conditions for which the operator (7.31) with domain (7.32) generates a contraction semigroup. The following technical lemma is useful.

Lemma 7.2.3. *Let Z be a Hilbert space with inner product $\langle \cdot,\cdot \rangle$, and let $\mathcal{P} \in \mathcal{L}(Z)$ be a* coercive *operator on Z, i.e., $\mathcal{P}$ is self-adjoint and $\mathcal{P} > \varepsilon I$, for some $\varepsilon > 0$.*

We define $Z_\mathcal{P}$ as the Hilbert space Z with the inner product $\langle \cdot,\cdot \rangle_\mathcal{P} := \langle \cdot, \mathcal{P}\cdot \rangle$. Then the operator A with domain $D(A)$ generates a contraction semigroup on Z if and only if $A\mathcal{P}$ with domain $D(A\mathcal{P}) = \{z \in Z \mid \mathcal{P}z \in D(A)\}$ generates a contraction semigroup on $Z_\mathcal{P}$.

Proof. By the coercivity of $\mathcal{P}$ it is easy to see that $Z_\mathcal{P}$ is a Hilbert space.

Since the inverse of $\mathcal{P}$ is also coercive, we only have to prove one implication. We assume that A with domain $D(A)$ generates a contraction semigroup.

We first show that $\mathrm{Re}\langle x, A\mathcal{P}x \rangle_\mathcal{P} \leq 0$ for $x \in D(A\mathcal{P})$. For $x \in D(A\mathcal{P})$, we have that

$$\mathrm{Re}\langle x, A\mathcal{P}x \rangle_\mathcal{P} = \mathrm{Re}\langle x, \mathcal{P}A\mathcal{P}x \rangle = \mathrm{Re}\langle \mathcal{P}x, A\mathcal{P}x \rangle \leq 0,$$

where we have used that $\mathcal{P}$ is self-adjoint and A is dissipative. Clearly, $A\mathcal{P}$ is a linear, densely defined and closed operator on $Z_\mathcal{P}$. Using Theorem 6.1.8 it remains to show that the adjoint of $A\mathcal{P}$ is dissipative as well. $z \in Z_\mathcal{P}$ is in the domain of the adjoint of $A\mathcal{P}$ if and only if there exists a $w \in Z_\mathcal{P}$ such that

$$\langle z, A\mathcal{P}x \rangle_\mathcal{P} = \langle w, x \rangle_\mathcal{P} \tag{7.33}$$

for all $x \in D(A\mathcal{P})$. Equation (7.33) is equivalent to

$$\langle \mathcal{P}z, A\mathcal{P}x \rangle = \langle w, \mathcal{P}x \rangle. \tag{7.34}$$

$x \in D(A\mathcal{P})$ if and only if $\mathcal{P}x \in D(A)$, which implies that (7.34) holds for all $x \in D(A\mathcal{P})$ if and only if

$$\langle \mathcal{P}z, A\tilde{x} \rangle = \langle w, \tilde{x} \rangle \tag{7.35}$$

holds for all $\tilde{x} \in D(A)$. Equation (7.35) is equivalent to the fact that $\mathcal{P}z \in D(A^*)$ and $w = A^*\mathcal{P}z$.

Summarizing, $z \in D((A\mathcal{P})^*)$ if and only if $\mathcal{P}z \in D(A^*)$ and furthermore,

$$(A\mathcal{P})^* z = A^*\mathcal{P}z, \qquad z \in D\left((A\mathcal{P})^*\right). \tag{7.36}$$

It remains to show that the adjoint as defined in (7.36) is dissipative. This is straightforward by using (7.36)

$$\mathrm{Re}\langle x, (A\mathcal{P})^* x\rangle_{\mathcal{P}} = \mathrm{Re}\langle x, \mathcal{P}A^*\mathcal{P}x\rangle = \mathrm{Re}\langle \mathcal{P}x, A^*\mathcal{P}x\rangle \leq 0,$$

where we used the fact that A^* is dissipative. □

The following theorem characterizes the matrices W_B for which the operator A with domain (7.32) generates a contraction semigroup. The matrix Σ is defined in (7.29). For the proof of this theorem we need some results concerning matrices, which can be found in Section 7.3.

Theorem 7.2.4. *Consider the operator A defined in* (7.31) *and* (7.32) *associated to a port-Hamiltonian system, that is, the assumptions of Definition* 7.1.2 *are satisfied. Furthermore, W_B, or equivalently $\tilde{W}_B$, is an $n \times 2n$ matrix of rank n. Then the following statements are equivalent.*

1. *A is the infinitesimal generator of a contraction semigroup on X.*
2. *$\mathrm{Re}\,\langle Ax, x\rangle_X \leq 0$ for every $x \in D(A)$.*
3. *$W_B \Sigma W_B^* \geq 0$.*

Proof. The Lumer-Phillips Theorem shows that part 1 implies part 2.

We next show that part 2 implies part 3. Thus we assume that $\mathrm{Re}\,\langle Ax, x\rangle_X \leq 0$ for every $x \in D(A)$. Using the fact that A is a restriction of A_0, Lemma 7.2.1 implies $f_\partial^* e_\partial + e_\partial^* f_\partial \leq 0$ for every $x \in D(A)$. Furthermore, by Lemma 7.2.1 for every pair $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$, there exists a function $x \in D(A)$ with boundary effort $e_\partial = e$ and boundary flow $f_\partial = f$. Thus we have $f^*e + e^*f \leq 0$ for every pair $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$.

We write W_B as $W_B = \left[\begin{array}{cc} W_1 & W_2 \end{array}\right]$. If y lies in the kernel of $W_1 + W_2$, then $W_B \left[\begin{smallmatrix} y \\ y \end{smallmatrix}\right] = 0$, and thus $y^*y + y^*y \leq 0$, which implies $y = 0$. This shows that the matrix $W_1 + W_2$ is injective, and hence invertible. Defining $V := (W_1 + W_2)^{-1}(W_1 - W_2)$, we have

$$\left[\begin{array}{cc} W_1 & W_2 \end{array}\right] = \frac{1}{2}(W_1 + W_2)\left[\begin{array}{cc} I + V & I - V \end{array}\right].$$

Let $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$ be arbitrary. By Lemma 7.3.2 there exists a vector ℓ such that $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \ell$. This implies

$$0 \geq f^*e + e^*f = \ell^*(-2I + 2V^*V)\ell, \tag{7.37}$$

This inequality holds for any $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$. Since the $n \times 2n$ matrix W_B has rank n, its kernel has dimension n, and so the set of vectors ℓ satisfying $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] = \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \ell$ for some $\left[\begin{smallmatrix} f \\ e \end{smallmatrix}\right] \in \ker W_B$ equals the whole space $\mathbb{K}^n$. Thus (7.37) implies that $V^*V \leq I$, and by Lemma 7.3.1 we obtain $W_B \Sigma W_B^* \geq 0$.

Finally, we show that part 3 implies part 1. The proof of this implication is divided in several steps. Using Lemma 7.3.1, we write the matrix W_B as $W_B = S\begin{bmatrix} I+V & I-V \end{bmatrix}$, where S is invertible and $V^*V \le I$.

Step 1. Using (7.31) and (7.32) we write $Ax = A_I \mathcal{H} x$, where A_I is defined by (7.31) and (7.32) with $\mathcal{H} \equiv I$. Furthermore, the inner product of the state space X, see (7.7), equals $\langle f, \mathcal{H} g\rangle_I$, where $\langle \cdot, \cdot \rangle_I$ is the inner product (7.7) with $\mathcal{H} \equiv I$, i.e., the standard inner product on $L^2([a,b];\mathbb{K}^n)$. Since $\mathcal{H}$ is a coercive multiplication operator on $L^2([a,b];\mathbb{K}^n)$, we may apply Lemma 7.2.3 to conclude that A generates a contraction semigroup on X if and only if A_I generates a contraction semigroup on $L^2([a,b];\mathbb{K}^n)$. Hence, it is sufficient to prove this implication for $\mathcal{H} = I$.

Step 2. Let $\mathcal{H} = I$ and $x \in D(A)$. Lemma 7.2.1 implies

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \frac{1}{2}\left(f_\partial^* e_\partial + e_\partial^* f_\partial\right).$$

By assumption, the vector $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ lies in the kernel of W_B. Using Lemma 7.3.2, $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ equals $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]\ell$ for some $\ell \in \mathbb{K}^n$. Thus we get

$$\begin{aligned}
\langle Ax, x\rangle_X + \langle x, Ax\rangle_X &= \frac{1}{2}\left(f_\partial^* e_\partial + e_\partial^* f_\partial\right) \\
&= \frac{1}{2}\left(\ell^*(I - V^*)(-I - V)\ell + \ell^*(-I - V^*)(I - V)\ell\right) \\
&= \ell^*(-I + V^*V)\ell \le 0,
\end{aligned} \tag{7.38}$$

where we used again Lemma 7.3.1. This implies the dissipativity of A.

Step 3. By the Lumer-Phillips Theorem (Theorem 6.1.7) it remains to show that the range of $I - A$ equals X.

The equation $(I - A)x = y$ is equivalent to the differential equation

$$x(\zeta) - P_1 \dot{x}(\zeta) - P_0 x(\zeta) = y(\zeta), \qquad \zeta \in [a,b]. \tag{7.39}$$

Thanks to the invertibility of the matrix P_1, the solution of (7.39) is given by

$$x(\zeta) = e^{(P_1^{-1} - P_1^{-1}P_0)(\zeta - a)} x(a) - \int_a^\zeta e^{(P_1^{-1} - P_1^{-1}P_0)(\zeta-\tau)} P_1^{-1} y(\tau) d\tau. \tag{7.40}$$

$x \in D(A)$ if and only if, see (7.26) and (7.30),

$$0 = W_B R_0 \begin{bmatrix} x(b) \\ x(a) \end{bmatrix} = W_B R_0 \begin{bmatrix} Ex(a) + q \\ x(a) \end{bmatrix}, \tag{7.41}$$

where $E = e^{(P_1^{-1} - P_1^{-1}P_0)(b-a)}$ and $q = -\int_a^b e^{(P_1^{-1} - P_1^{-1}P_0)(b-\tau)} P_1^{-1} y(\tau) d\tau$, see (7.40). Equation (7.41) can be equivalently written as

$$W_B R_0 \begin{bmatrix} E \\ I \end{bmatrix} x(a) = -W_B R_0 \begin{bmatrix} q \\ 0 \end{bmatrix}. \tag{7.42}$$

Next we prove that the square matrix $W_B R_0 \left[\begin{smallmatrix} E \\ I \end{smallmatrix}\right]$ is invertible. A square matrix is invertible if and only if it is injective. Therefore, we assume that there exists a vector $r_0 \neq 0$ such that

$$W_B R_0 \begin{bmatrix} E \\ I \end{bmatrix} r_0 = 0. \tag{7.43}$$

Now we solve equation (7.39) for $x(a) = r_0$ and $y \equiv 0$. The corresponding solution lies in $D(A)$ by (7.40) and (7.41). Thus $(I - A)x = 0$ has a non-zero solution. In other words, x, being the solution of (7.39) with $x(a) = r_0$ and $y \equiv 0$, is an eigenfunction of A with eigenvalue one. However, by (7.38), A possesses no eigenvalues in the right half-plane, which leads to a contradiction, implying that $W_B R_0 \left[\begin{smallmatrix} E \\ I \end{smallmatrix}\right]$ is invertible, and so (7.41) has a unique solution in the domain of A. The function y was arbitrary, and so we have proved that the range of $I - A$ equals X. Using the Lumer-Phillips Theorem 6.1.7, we conclude that A generates a contraction semigroup. □

We apply this theorem to our vibrating string example, see Example 7.1.1.

Example 7.2.5. Consider the vibrating string on the spatial domain $[a, b]$, see Example 7.1.1. The string is fixed at the left-hand side and we apply no force at the right-hand side, which gives the boundary conditions

$$\frac{\partial w}{\partial t}(a,t) = 0, \qquad T(b)\frac{\partial w}{\partial \zeta}(b,t) = 0. \tag{7.44}$$

We have $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $P_0 = 0$, and $\mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}$, see (7.5). Using this and equation (7.23) the boundary variables are given by

$$f_\partial = \frac{1}{\sqrt{2}} \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \end{bmatrix}, \quad e_\partial = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a) \end{bmatrix}. \tag{7.45}$$

The boundary condition (7.44) becomes in these variables

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b,t) \\ \frac{\partial w}{\partial t}(a,t) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \end{aligned} \tag{7.46}$$

with $W_B = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{smallmatrix}\right]$. Since W_B is a 2×4 matrix with rank 2, and since $W_B \Sigma W_B^T = 0$, we conclude from Theorem 7.2.4 that the operator associated to the p.d.e. generates a contraction semigroup on $L^2([a, b]; \mathbb{R}^2)$ with the norm (7.6).

In the above we used part 3 of Theorem 7.2.4 to show that the operator associated to the p.d.e. (7.1) with boundary conditions (7.44) generates a contraction semigroup on the energy space. To complete this example we show how this can be proved using part 2 of Theorem 7.2.4.

From (7.25) and (7.44) we see that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b,t) \\ \frac{\partial w}{\partial t}(a,t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(b) \end{bmatrix} = \tilde{W}_B \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(b) \end{bmatrix}. \tag{7.47}$$

It is clear that the rank of $\tilde{W}_B$ is two. The operator A is given as

$$Ax = P_1 \frac{d}{d\zeta}(\mathcal{H}x) = \begin{bmatrix} \frac{d}{d\zeta}(Tx_2) \\ \frac{d}{d\zeta}\left(\frac{x_1}{\rho}\right) \end{bmatrix} \tag{7.48}$$

with domain

$$D(A) = \{x \in L^2([a,b];\mathbb{R}^2) \mid \mathcal{H}x \in H^1([a,b];\mathbb{R}^2), \text{ and (7.47) holds}\}.$$

Thus for $x \in D(A)$ we have

$$\begin{aligned} \langle Ax, x\rangle &= \int_a^b \frac{d}{d\zeta}(Tx_2)(\zeta)\cdot\frac{x_1}{\rho}(\zeta) + \frac{d}{d\zeta}\left(\frac{x_1}{\rho}\right)(\zeta)\cdot(Tx_2)(\zeta)d\zeta \\ &= \left[(Tx_2)(\zeta)\frac{x_1}{\rho}(\zeta)\right]_a^b = 0 - 0 = 0, \end{aligned}$$

where we used the boundary conditions (7.47). By part 2 of Theorem 7.2.4, we conclude that A generates a contraction semigroup on X.

7.3 Technical lemmas

This section contains two technical lemmas on matrix representations. They are important for the proof of Theorem 7.2.4, but not for the understanding of the examples.

Lemma 7.3.1. *Let W be a $n \times 2n$ matrix and let $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$. Then W has rank n and $W\Sigma W^* \geq 0$ if and only if there exist a matrix $V \in \mathbb{K}^{n\times n}$ and an invertible matrix $S \in \mathbb{K}^{n\times n}$ such that*

$$W = S\begin{bmatrix} I+V & I-V \end{bmatrix} \tag{7.49}$$

with $VV^* \leq I$, or equivalently $V^*V \leq I$.

Proof. Sufficiency: If W is of the form (7.49), then we find

$$W\Sigma W^* = S\begin{bmatrix} I+V & I-V \end{bmatrix}\Sigma\begin{bmatrix} I+V^* \\ I-V^* \end{bmatrix}S^* = S(2I-2VV^*)S^*, \tag{7.50}$$

which is non-negative, since $VV^* \leq I$. Assuming $\operatorname{rk} W < n$, there exists $x \neq 0$ such that $x^*W = 0$, or equivalently $(S^*x)^*\begin{bmatrix} I+V & I-V \end{bmatrix} = 0$. This implies for $\tilde{x} := S^*x$ that $\tilde{x}(I+V) = 0$ and $\tilde{x}(I-V) = 0$, and thus $\tilde{x} = 0$, which leads to a contradiction. Therefore, $\operatorname{rk} W = n$.

Necessity: Writing W as $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$, we see that $W\Sigma W^* \geq 0$ is equivalent to $W_1 W_2^* + W_2 W_1^* \geq 0$. Hence

$$(W_1 + W_2)(W_1 + W_2)^* \geq (W_1 - W_2)(W_1 - W_2)^* \geq 0. \tag{7.51}$$

If $x \in \ker((W_1+W_2)^*)$, then the above inequality implies that $x \in \ker((W_1-W_2)^*)$. Thus $x \in \ker(W_1^*) \cap \ker(W_2^*)$. Since W has full rank, this implies that $x = 0$. Hence $W_1 + W_2$ is invertible. Using (7.51) once more, we see that

$$(W_1 + W_2)^{-1}(W_1 - W_2)(W_1 - W_2)^*(W_1 + W_2)^{-*} \leq I$$

and thus $V := (W_1 + W_2)^{-1}(W_1 - W_2)$ satisfies $VV^* \leq I$. Summarizing, we have

$$\begin{aligned} \begin{bmatrix} W_1 & W_2 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} W_1 + W_2 + W_1 - W_2 & W_1 + W_2 - W_1 + W_2 \end{bmatrix} \\ &= \frac{1}{2}(W_1 + W_2) \begin{bmatrix} I+V & I-V \end{bmatrix}. \end{aligned}$$

Defining $S := \frac{1}{2}(W_1 + W_2)$, we have shown the representation (7.49). □

Lemma 7.3.2. *Suppose that the $n \times 2n$ matrix W can be written in the format of equation* (7.49)*, i.e., $W = S[I+V \;\; I-V]$ with S and V square matrices, and S invertible. Then the kernel of W equals the range of $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$.*

Proof. Let $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ be in the range of $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$. By the equality (7.49), we have that

$$\begin{aligned} W \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= S \begin{bmatrix} I+V & I-V \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= S \begin{bmatrix} I+V & I-V \end{bmatrix} \begin{bmatrix} I-V \\ -I-V \end{bmatrix} \ell = 0. \end{aligned}$$

Hence the range of $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$ lies in the kernel of W. The conditions on W imply that $\operatorname{rk} W = n$, and so the kernel of W has dimension n. It is sufficient to show that the $2n \times n$-matrix $\left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right]$ has full rank. If this is not the case, then there is a non-trivial element in its kernel. It is easy to see that the kernel consists of zero only, and thus we have proved the lemma. □

7.4 Exercises

7.1. Consider the *transmission line* on the spatial interval $[a, b]$:

$$\begin{aligned} \frac{\partial Q}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta} \frac{\phi(\zeta, t)}{L(\zeta)}, \\ \frac{\partial \phi}{\partial t}(\zeta, t) &= -\frac{\partial}{\partial \zeta} \frac{Q(\zeta, t)}{C(\zeta)}. \end{aligned} \tag{7.52}$$

Here $Q(\zeta, t)$ is the charge at position $\zeta \in [a, b]$ and time $t > 0$, and $\phi(\zeta, t)$ is the (magnetic) flux at position ζ and time t. C is the (distributed) capacity and L is the (distributed) inductance.

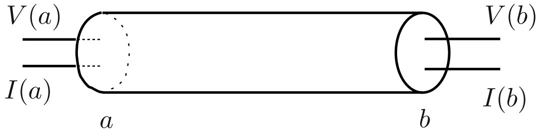

Figure 7.2: Transmission line

The voltage and current are given by $V = Q/C$ and $I = \phi/L$, respectively. The energy of this system is given by

$$E(t) = \frac{1}{2}\int_a^b \frac{\phi(\zeta, t)^2}{L(\zeta)} + \frac{Q(\zeta, t)^2}{C(\zeta)} d\zeta. \tag{7.53}$$

Formulate the transmission line as depicted in Figure 7.2 as a port-Hamiltonian system, see Definition 7.1.2.

7.2. Consider the operator A defined by (7.31) and (7.32) associated to a port-Hamiltonian system, that is, the assumptions of Definition 7.1.2 are satisfied. Furthermore, assume that W_B is a full rank $n \times 2n$ matrix.

Show that the following are equivalent:

(a) A is the infinitesimal generator of a unitary group on X.

(b) $\text{Re}\langle Ax, x\rangle_X = 0$ for all $x \in D(A)$.

(c) $W_B \Sigma W_B^* = 0$.

7.3. Let $L^2([a, b]; \mathbb{K}^n)$ be the standard Lebesgue space with its standard inner product.

Consider the operator A defined by (7.31) and (7.32) with W_B a full rank $n \times 2n$ matrix satisfying $W_B \Sigma W_B^* \geq 0$. Further we assume that the assumptions of Definition 7.1.2 are satisfied. Show that A generates a strongly continuous semigroup on $L^2([a, b]; \mathbb{K}^n)$. Is this semigroup a contraction semigroup?

7.4. Consider coupled vibrating strings as given in the figure below. We assume that the length of all strings are equal. The model for every vibrating string is given by (1.13) with physical parameters, $\rho_{\text{I}}, T_{\text{I}}, \rho_{\text{II}}$, etc. Furthermore, we assume that the three strings are connected via a (mass-less) bar, as shown in Figure 7.3. This bar can only move in the vertical direction. This implies that the velocity of string I at its right-hand side equals those of the other two strings at their left-hand side. Furthermore, the force of string I at its

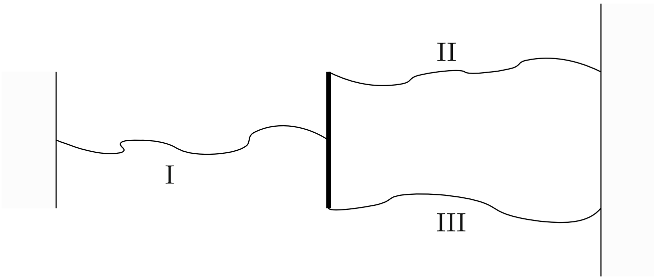

Figure 7.3: Coupled vibrating strings

right-end side equals the sum of the forces of the other two at their left-hand side, i.e.,

$$T_{\mathrm{I}}(b)\frac{\partial w_{\mathrm{I}}}{\partial \zeta}(b) = T_{\mathrm{II}}(a)\frac{\partial w_{\mathrm{II}}}{\partial \zeta}(a) + T_{\mathrm{III}}(a)\frac{\partial w_{\mathrm{III}}}{\partial \zeta}(a).$$

As depicted, the strings are attached to a wall.

(a) Identify the boundary conditions for the system given in Figure 7.3.

(b) Formulate the coupled strings as a port-Hamiltonian system (7.17) and (7.25). Furthermore, determine the energy space X.

(c) Show that the differential operator associated to the above system generates a contraction semigroup on the energy space X.

7.5. In this exercise we show that a second order differential equation associated to a *Sturm-Liouville operator* determines a port-Hamiltonian system. The Sturm-Liouville operator is the differential operator A_{SL} on $L^2(a,b)$ given by

$$A_{SL}h = \frac{1}{w}\left(-\frac{d}{d\zeta}(p\frac{dh}{d\zeta}) + qh\right),$$

where $w \in C([a,b])$, $p \in C^1([a,b])$ and $q \in C([a,b])$ are real-valued functions with $p(\zeta) > 0$ and $w(\zeta) > 0$ for every $\zeta \in [a,b]$. We define the domain of A_{SL} as follows:

$$\begin{aligned} D(A_{SL}) \quad = \quad & \Big\{ h \in L^2(a,b) \mid h, \frac{dh}{d\zeta} \text{ are absolutely continuous,} \\ & \frac{d^2h}{d\zeta^2} \in L^2(a,b), \beta_1 h(a) + \gamma_1 \frac{dh}{dx}(a) = 0, \text{ and} \\ & \beta_2 h(b) + \gamma_2 \frac{dh}{dx}(b) = 0 \Big\}, \end{aligned}$$

where we suppose that $\beta_1, \beta_2, \gamma_1$, and γ_2 are real constants satisfying $|\beta_1| + |\gamma_1| > 0$, and $|\beta_2| + |\gamma_2| > 0$.

Assume now that q is identically zero.

(a) Write the second order differential equation

$$\frac{\partial^2 h}{\partial t^2} = -A_{SL} h$$

as a port-Hamiltonian system. What is the energy space?
Hint: See Example 7.1.1.

(b) Determine necessary and sufficient conditions on the constants β_1, β_2, γ_1, and γ_2 such that A_{SL} generates a contraction semigroup on the energy space.

7.5 Notes and references

In this chapter we closely follow the article [36] by Le Gorrec, Zwart, and Maschke, although we present a new proof of Theorem 7.2.4. Strangely enough the class of linear, first order port-Hamiltonian systems has not been studied before within the infinite-dimensional system theory community. This class was introduced using the language of differential forms in van der Schaft and Maschke [57]. Although the class of linear, first order port-Hamiltonian systems was not studied within systems theory, one may find results on a strongly related class of operators in the literature, for instance in [19]. In [19], the authors are mainly interested in self-adjoint extensions of symmetric operators, that is, the question is to determine those boundary conditions for which an operator is self-adjoint. Note that the multiplication of a self-adjoint operator by i is a skew-adjoint operator. Skew-adjoint operators are the infinitesimal generators of unitary semigroups, see Remark 6.2.6. In order to apply results from [19] we have to multiply the operators by i. The result formulated in Lemma 7.2.3 for Banach spaces can be traced back to Gustafson and Lumer [20]. However, we present a more direct proof for the Hilbert space situation.

Chapter 8

Stability

This chapter is devoted to stability of abstract differential equations as well as to spectral projections and invariant subspaces. One of the most important aspects of systems theory is stability, which is closely connected to the design of feedback controls. For infinite-dimensional systems there are different notions of stability such as strong stability, polynomial stability, and exponential stability. In this chapter, we restrict ourselves to exponential stability. Strong stability will be defined in an exercise, where it is also shown that strong stability is weaker than exponential stability. The concept of invariant subspaces, which we discuss in the second part of this chapter will play a key role in the study of stabilizability in Chapter 10.

8.1 Exponential stability

We start with the definition of exponential stability.

Definition 8.1.1. The C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X is *exponentially stable* if there exist positive constants M and α such that

$$\|T(t)\| \leq Me^{-\alpha t} \qquad \text{for } t \geq 0. \tag{8.1}$$

The constant α is called the *decay rate*, and the supremum over all possible values of α is the *stability margin* of $(T(t))_{t\geq 0}$; this is minus its growth bound (see Theorem 5.1.5).

If $(T(t))_{t\geq 0}$ is exponentially stable, then the solution to the abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \; t \geq 0, \qquad x(0) = x_0, \tag{8.2}$$

tends to zero exponentially fast as $t \to \infty$. Datko's lemma is an important criterion for exponential stability.

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Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_8,

Lemma 8.1.2 (Datko's lemma)**.** *The C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X is exponentially stable if and only if for every $x \in X$ there holds*

$$\int_0^\infty \|T(t)x\|^2 \, dt < \infty. \tag{8.3}$$

Proof. The necessity is obvious, so suppose that (8.3) holds. Now Theorem 5.1.5 implies that there exist numbers $M > 0$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for } t \geq 0. \tag{8.4}$$

Thus for every $n \in \mathbb{N}$ the operator Q_n, defined by

$$(Q_n x)(t) := \ {}_{[0,n]}(t)T(t)x,$$

is a bounded linear operator from X to $L^2([0,\infty); X)$. (8.3) implies that for every $x \in X$ the family $\{Q_n x, n \in \mathbb{N}\}$ is uniformly bounded in n, and thus by the Uniform Boundedness Theorem, it follows that

$$\|Q_n\| \leq \gamma \tag{8.5}$$

for some γ independent of n.

For $0 \leq t \leq 1$, we have that $\|T(t)\| \leq Me^{\omega t} \leq Me^{\omega}$. For $t > 1$, we calculate

$$\begin{aligned}
\frac{1 - e^{-2\omega t}}{2\omega}\|T(t)x\|^2 &= \int_0^t e^{-2\omega s}\|T(t)x\|^2 ds \\
&\leq \int_0^t e^{-2\omega s}\|T(s)\|^2\|T(t-s)x\|^2 ds \\
&\leq M^2 \int_0^t \|T(t-s)x\|^2 ds \qquad \text{using (8.4)} \\
&= M^2 \int_0^t \|T(s)x\|^2 ds \leq M^2\gamma^2\|x\|^2,
\end{aligned}$$

where we used (8.5). Thus for some $K > 0$ and all $t \geq 0$, we obtain

$$\|T(t)\| \leq K$$

and, moreover,

$$\begin{aligned}
t\|T(t)x\|^2 &= \int_0^t \|T(t)x\|^2 ds \\
&\leq \int_0^t \|T(s)\|^2\|T(t-s)x\|^2 ds \leq K^2\gamma^2\|x\|^2 \qquad \text{using (8.5).}
\end{aligned}$$

Hence

$$\|T(t)\| \leq \frac{K\gamma}{\sqrt{t}},$$

which implies that $\|T(\tau)\| < 1$ for a sufficiently large τ. Consequently, $\log(\|T(\tau)\|) < 0$, and thus by Theorem 5.1.5 there exist $\tilde{M}$, $\alpha > 0$ such that

$$\|T(t)\| \leq \tilde{M}e^{-\alpha t} \qquad \text{for all } t \geq 0. \qquad \square$$

Lemma 8.1.2 can be used to prove a Lyapunov-type result, which will be of use in establishing stability of the abstract differential equation.

Theorem 8.1.3. *Suppose that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. Then the following are equivalent*

1. $(T(t))_{t\geq 0}$ *is exponentially stable;*
2. *There exists a positive operator $P \in \mathcal{L}(X)$ such that*

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\langle x, x\rangle \qquad \text{for all } x \in D(A). \tag{8.6}$$

3. *There exists a positive operator $P \in \mathcal{L}(X)$ such that*

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle \leq -\langle x, x\rangle \qquad \text{for all } x \in D(A). \tag{8.7}$$

Equation (8.6) *is called a* Lyapunov equation.

Proof. 1 $\Rightarrow$ 2: Since $(T(t))_{t\geq 0}$ is exponentially stable, the following operator is well defined

$$\langle x_1, Px_2\rangle = \int_0^\infty \langle T(t)x_1, T(t)x_2\rangle\, dt. \tag{8.8}$$

Since

$$\begin{aligned}|\langle x_1, Px_2\rangle| &\leq \int_0^\infty \|T(t)x_1\|\|T(t)x_2\|dt \leq \int_0^\infty M^2e^{-2\alpha t}\|x_1\|\|x_2\|dt\\ &= \frac{M^2}{2\alpha}\|x_1\|\|x_2\|,\end{aligned}$$

P is bounded. Furthermore,

$$\langle x, Px\rangle = \int_0^\infty \|T(s)x\|^2 ds \geq 0$$

and $\langle x, Px\rangle = 0$ implies that $\|T(t)x\| = 0$ on $[0,\infty)$ almost everywhere. The strong continuity of $(T(t))_{t\geq 0}$ implies that $x = 0$. Thus $P > 0$. It remains to show that P satisfies (8.6). Using (8.8), we have for $x \in D(A)$

$$\begin{aligned}\langle Ax, Px\rangle + \langle Px, Ax\rangle &= \int_0^\infty \langle T(t)Ax, T(t)x\rangle + \langle T(t)x, T(t)Ax\rangle\, dt\\ &= \int_0^\infty \frac{d\langle T(t)x, T(t)x\rangle}{dt}\, dt = 0 - \langle x, x\rangle,\end{aligned}$$

where we have used the exponential stability of $(T(t))_{t\geq 0}$. Thus P defined by (8.8) is a solution of (8.6).

$2 \Rightarrow 3$ is trivial, and thus it remains to prove $3 \Rightarrow 1$: Suppose that there exists a bounded $P > 0$ such that (8.7) is satisfied. We introduce the following Lyapunov functional:

$$V(t,x) = \langle PT(t)x, T(t)x\rangle.$$

Since P is positive, $V(t,x) \geq 0$ for all $t \geq 0$. For $x \in D(A)$, we may differentiate V to obtain by (8.7)

$$\frac{dV}{dt}(t,x) = \langle PAT(t)x, T(t)x\rangle + \langle PT(t)x, AT(t)x\rangle \leq -\|T(t)x\|^2.$$

Integration yields

$$0 \leq V(t,x) \leq V(0,x) - \int_0^t \|T(s)x\|^2 ds$$

and hence

$$\int_0^t \|T(s)x\|^2 ds \leq V(0,x) = \langle Px, x\rangle \qquad \text{for all } t \geq 0 \text{ and } x \in D(A).$$

This inequality can be extended to all $x \in X$, since $D(A)$ is dense in X. In other words, for every $x \in X$ we have that

$$\int_0^\infty \|T(s)x\|^2 ds \leq \langle Px, x\rangle < \infty$$

and Datko's Lemma 8.1.2 completes the proof. □

In finite dimensions, one usually examines exponential stability via the spectrum of the operator. However, this is not feasible in infinite dimensions. While the inequality

$$\sup_{\lambda\in\sigma(A)} \operatorname{Re}(\lambda) \leq \lim_{t\to\infty} \frac{\log\|T(t)\|}{t} = \omega_0 \tag{8.9}$$

always holds (Proposition 5.2.4), one need not necessarily have equality. For an example we refer the reader to [62] or [10, Example 5.1.4]. Although the location of the spectrum is not sufficient to determine the stability, the uniform boundedness of the resolvent operator is sufficient.

Theorem 8.1.4. *Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. Then $(T(t))_{t\geq 0}$ is exponentially stable if and only if $(\cdot I - A)^{-1} \in \mathbf{H}^\infty(\mathcal{L}(X))$.*

The definition and properties of the spaces $\mathbf{H}^\infty(\mathcal{L}(X))$ and $\mathbf{H}^2(X)$ can be found in Appendix A.2.

Proof. Necessity. By assumption, we know that the C_0-semigroup satisfies $\|T(t)\| \leq Me^{\omega t}$ for some $\omega < 0$. Proposition 5.2.4 implies that $\overline{\mathbb{C}_0^+} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$, the closed right-half plane, is contained in the resolvent set of A and, furthermore, for $s \in \overline{\mathbb{C}_0^+}$

$$\|(sI - A)^{-1}\| \leq \frac{M}{\mathrm{Re}(s) - \omega} \leq \frac{M}{-\omega}.$$

Combining this with Proposition 5.2.4.3, we conclude that $(\cdot I - A)^{-1} \in \mathbf{H}^\infty(\mathcal{L}(X))$.

Sufficiency. Suppose that the C_0-semigroup satisfies $\|T(t)\| \leq Me^{(\omega-\varepsilon)t}$ for some positive constants M, ω and ε. It is easy to see that $e^{-\omega \cdot}T(\cdot)x$ is an element of $L^2([0,\infty); X)$ for every $x \in X$. Furthermore, the Laplace transform of $t \mapsto e^{-\omega t}T(t)x$ equals $s \mapsto ((s+\omega)I - A)^{-1}x$ (see Proposition 5.2.4). The Paley-Wiener Theorem A.2.9 implies

$$((\cdot + \omega)I - A)^{-1}x \in \mathbf{H}^2(X).$$

Now, by assumption, $(\cdot I - A)^{-1} \in \mathbf{H}^\infty(\mathcal{L}(X))$, and by Theorem A.2.10.2 it follows that

$$(\cdot I - A)^{-1}((\cdot + \omega)I - A)^{-1}x \in \mathbf{H}^2(X).$$

Using the resolvent equation, we conclude that $(\cdot I - A)^{-1}x \in \mathbf{H}^2(X)$, since

$$(\cdot I - A)^{-1}x = ((\cdot + \omega)I - A)^{-1}x + \omega(\cdot I - A)^{-1}((\cdot + \omega)I - A)^{-1}x. \qquad (8.10)$$

However, the Laplace transform of $t \mapsto T(t)x$ is $s \mapsto (sI - A)^{-1}x$ and thus by the Paley-Wiener Theorem A.2.9, we have that $T(\cdot)x \in L^2([0,\infty); X)$. Finally, Lemma 8.1.2 shows that $(T(t))_{t\geq 0}$ is exponentially stable. □

8.2 Spectral projection and invariant subspaces

In this section, we discuss various invariance concepts and study the relationships between them. First we define $T(t)$-invariance.

Definition 8.2.1. Let V be a subspace of the Hilbert space X and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. We say that V is $T(t)$-*invariant* if for all $t \geq 0$

$$T(t)V \subset V.$$

Since $x(t) = T(t)x_0$ is the solution of (8.2) we see that $T(t)$-invariance is equivalent to the fact that the solution stays of (8.2) in V when the initial condition lies in V.

Definition 8.2.2. Let V be a subspace of the Hilbert space X and let A be an infinitesimal generator of a C_0-semigroup on X. We say that V is A-*invariant* if

$$A(V \cap D(A)) \subset V.$$

For finite-dimensional systems, it is well known that a (closed) subspace is $T(t)$-invariant if and only if it is A-invariant, where A is the infinitesimal generator of $T(t) = e^{At}$, $t \geq 0$. This result generalizes to bounded generators A, see Exercise 8.4. However, this results does in general not hold for infinite-dimensional systems, see [63]. However, we do have the following partial result.

Lemma 8.2.3. *Let V be a closed subspace of X and let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. If V is $T(t)$-invariant, then V is A-invariant.*

Proof. Let v be an arbitrary element in $V \cap D(A)$. Then Definition 5.2.1 implies

$$\lim_{t\downarrow 0} \frac{1}{t}(T(t) - I)v = Av.$$

By assumption, $\frac{1}{t}(T(t) - I)v$ is an element of V for every $t > 0$. Thus, since V is closed, the limit is also an element of V and therefore $Av \in V$. □

Whether or not a C_0-semigroup has a nontrivial $T(t)$-invariant subspace is a fundamental question. If the spectrum of A consists of two or more regions, then this question can be answered positively, as shown in the following theorem.

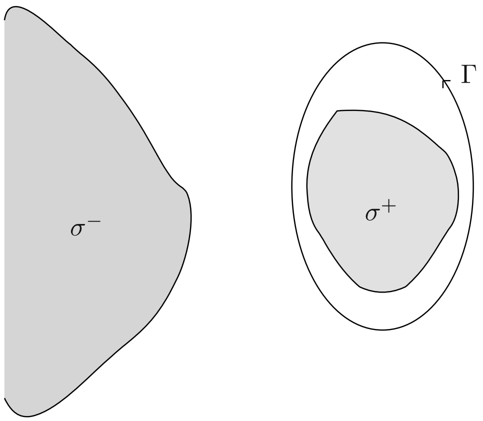

Figure 8.1: Spectral decomposition

Theorem 8.2.4. *Let A be a closed densely defined operator on X. Assume that the spectrum of A is the union of two parts, σ^+ and σ^-, such that a rectifiable, closed, simple curve Γ can be drawn so as to enclose an open set containing σ^+ in its interior and σ^- in its exterior. The operator, P_Γ, defined by*

$$P_\Gamma x = \frac{1}{2\pi i} \int_\Gamma (\lambda I - A)^{-1} x d\lambda, \tag{8.11}$$

where Γ *is traversed once in the positive direction (counterclockwise), is a projection. We call this projection the* spectral projection *on* σ^+. *This projection induces a decomposition of the state space*

$$X = X^+ \oplus X^-, \text{ where } X^+ = P_\Gamma X, \text{ and } X^- = (I - P_\Gamma)X. \tag{8.12}$$

Moreover, the following properties hold:

1. *For* $x \in D(A)$ *we have* $P_\Gamma Ax = AP_\Gamma x$. *Furthermore, for all* $s \in \rho(A)$ *we have* $(sI - A)^{-1}P_\Gamma = P_\Gamma(sI - A)^{-1}$.
2. *The spaces* X^+ *and* X^- *are* A*-invariant, and for all* $s \in \rho(A)$ *there holds that* $(sI - A)^{-1}X^+ \subset X^+$ *and* $(sI - A)^{-1}X^- \subset X^-$;
3. $P_\Gamma X \subset D(A)$, *and the restriction of* A *to* X^+, A^+, *is a bounded operator on* X^+;
4. $\sigma(A^+) = \sigma^+$ *and* $\sigma(A^-) = \sigma^-$, *where* A^- *is the restriction of* A *to* X^-. *Furthermore, for* $\lambda \in \rho(A)$ *we have that* $(\lambda I - A^+)^{-1} = (\lambda I - A)^{-1}|_{X^+}$ *and* $(\lambda I - A^-)^{-1} = (\lambda I - A)^{-1}|_{X^-}$;
5. *If* σ^+ *consists of only finitely many eigenvalues with finite order, then* P_Γ *projects onto the space of generalized eigenvectors of the enclosed eigenvalues. Thus we have that*

   $$\operatorname{ran} P_\Gamma = \sum_{\lambda_n \in \sigma^+} \ker(\lambda_n I - A)^{\nu(n)} = \sum_{\lambda_n \in \sigma^+} \ker(\lambda_n I - A^+)^{\nu(n)},$$

   where $\nu(n)$ *is the order of* λ_n;
6. *If* $\sigma^+ = \{\lambda_n\}$ *with* λ_n *an eigenvalue of multiplicity* 1, *then*

   $$P_\Gamma z = \langle z, \psi_n \rangle \phi_n,$$

   where ϕ_n *is the eigenvector of* A *corresponding to* λ_n *and* ψ_n *is an eigenvector of* A^* *corresponding to* $\overline{\lambda_n}$ *with* $\langle \phi_n, \psi_n \rangle = 1$.

The order of an isolated eigenvalue λ_0 is defined as follows. We say that λ_0 has *order* ν_0 if for every $x \in X$ $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0} (\lambda I - A)^{-1} x$ exists, but there exists an x_0 such that the limit $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0 - 1} (\lambda I - A)^{-1} x_0$ does not exist. If for every $\nu \in \mathbb{N}$ there exists an $x_\nu \in X$ such that the limit $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu} (\lambda I - A)^{-1} x_\nu$ does not exist, then the order of λ_0 is infinity.

An eigenvalue λ has *multiplicity* 1, if the order is 1, and if $\dim \ker(\lambda I - A) = 1$.

Proof. Since the mapping $\lambda \mapsto (\lambda I - A)^{-1}$ is uniformly bounded on Γ, P_Γ is a bounded linear operator on X. For $s \in \rho(A)$ we have that

$$(sI - A)^{-1} P_\Gamma x = \frac{1}{2\pi i} \int_\Gamma (sI - A)^{-1} (\lambda I - A)^{-1} x d\lambda = P_\Gamma (sI - A)^{-1} x. \tag{8.13}$$

Using the resolvent identity for the middle term we find, for $s \in \rho(A)$ and outside Γ,

$$
\begin{aligned}
(sI - A)^{-1} P_\Gamma x &= \frac{1}{2\pi i} \int_\Gamma \frac{-(sI - A)^{-1}x}{s - \lambda} + \frac{(\lambda I - A)^{-1}x}{s - \lambda} d\lambda \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{(\lambda I - A)^{-1}x}{s - \lambda} d\lambda, \qquad (8.14)
\end{aligned}
$$

by Cauchy's residue theorem.

Using this relation we show that P_Γ is a projection. Let Γ' be another rectifiable, simple, closed curve enclosing σ^+ counterclockwise that encircles Γ as well. Then by standard complex analysis and the fact that $(sI - A)^{-1}$ is holomorphic between the two curves we have that P_Γ is also given by

$$
P_\Gamma x = \frac{1}{2\pi i} \int_{\Gamma'} (\lambda I - A)^{-1} x d\lambda.
$$

Hence, with (8.14) we obtain

$$
\begin{aligned}
P_\Gamma P_\Gamma x &= \frac{1}{2\pi i} \int_{\Gamma'} (sI - A)^{-1} P_\Gamma x ds \\
&= \frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{2\pi i} \int_\Gamma \frac{(\lambda I - A)^{-1}x}{s - \lambda} d\lambda ds \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{s - \lambda} ds (\lambda I - A)^{-1} x d\lambda \qquad \text{by Fubini's theorem} \\
&= \frac{1}{2\pi i} \int_\Gamma (\lambda I - A)^{-1} x d\lambda = P_\Gamma x,
\end{aligned}
$$

where we used Cauchy's theorem. Thus P_Γ is a projection. This immediately implies that $X^+ := P_\Gamma X$ and $X^- := (I - P_\Gamma)X$ are closed linear subspaces and $X = X^+ \oplus X^-$. Now we prove part 1 to 6.

Parts 1 *and* 2. By (8.13) it follows that P_Γ commutes with the resolvent operator, and hence also $AP_\Gamma = P_\Gamma A$ on the domain of A. So $X^+ = \operatorname{ran} P_\Gamma$ and $X^- = \operatorname{ran}(I - P_\Gamma)$ are A- and $(sI - A)^{-1}$-invariant, i.e., $(sI - A)^{-1} X^+ \subset X^+$, $(sI - A)^{-1} X^- \subset X^-$.

Part 3. We show that $P_\Gamma X \subset D(A)$. For $\lambda, s \in \rho(A)$ it holds that $(sI - A)(\lambda I - A)^{-1} = (s - \lambda)(\lambda I - A)^{-1} + I$. Therefore, for $x \in X$ we obtain

$$
\begin{aligned}
(sI - A)^{-1} &\frac{1}{2\pi i} \int_\Gamma (s - \lambda)(\lambda I - A)^{-1} x d\lambda \\
&= (sI - A)^{-1} \left(\frac{1}{2\pi i} \int_\Gamma (sI - A)(\lambda I - A)^{-1} x d\lambda - \frac{1}{2\pi i} \int_\Gamma x d\lambda \right) \\
&= \frac{1}{2\pi i} \int_\Gamma (sI - A)^{-1}(sI - A)(\lambda I - A)^{-1} x d\lambda = P_\Gamma x. \qquad (8.15)
\end{aligned}
$$

(8.15) holds for any $x \in X$, and so $X^+ = P_\Gamma X \subset D(A)$. Since X^+ is A-invariant, $A|_{X^+}$ is well-defined. $A|_{X^+}$ is closed, since A is closed. Now A^+ is defined on the whole space X^+, and thus by the closed graph theorem A^+ is bounded on X^+.

Part 4. Let s be an element of $\mathbb{C}$ that does not lie on Γ. We define

$$Q_s x := \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda I - A)^{-1}x}{s-\lambda}d\lambda.$$

It is easy to see that this operator commutes with the resolvent, and similarly as in part 1 and 2 we find that

$$Q_s P_\Gamma = P_\Gamma Q_s \qquad \text{and} \qquad AQ_s = Q_s A \quad \text{on } D(A). \tag{8.16}$$

From the first relation, we conclude that X^+ and X^- are Q_s-invariant.

For $s \neq \lambda$ we have that

$$\frac{1}{s-\lambda}(sI-A)(\lambda I - A)^{-1} = (\lambda I - A)^{-1} + \frac{1}{s-\lambda}I.$$

Thus for $x \in D(A)$ and $s \notin \Gamma$

$$(sI-A)Q_s x = \frac{1}{2\pi i}\int_\Gamma \frac{(sI-A)(\lambda I - A)^{-1}x}{s-\lambda}d\lambda = P_\Gamma x + \frac{1}{2\pi i}\int_\Gamma \frac{1}{s-\lambda}x d\lambda. \tag{8.17}$$

For s outside Γ and $x \in X^+$, we get

$$(sI - A^+)Q_s x = (sI-A)Q_s x = P_\Gamma x = x,$$

where in the first equality we have used that Q_s maps X^+ into X^+. Since Q_s commutes with A, we find

$$(sI - A^+)Q_s = Q_s(sI - A^+) = I_{X^+}. \tag{8.18}$$

Similar, we find, for s inside Γ,

$$(sI-A^-)Q_s = I_{X^-} \text{ and } Q_s(sI-A^-)x = x \quad \text{for } x \in D(A^-) = D(A)\cap X^-. \tag{8.19}$$

Thus any s outside Γ lies in the resolvent set of A^+, and any s inside Γ lies in the resolvent set of A^-.

On the other hand, the A- and $(sI-A)^{-1}$-invariance of X^+ and X^-, see part 1 and 2, gives that

$$\rho(A) = \rho(A^+) \cap \rho(A^-), \text{ and (equivalently) } \sigma(A) = \sigma(A^+) \cup \sigma(A^-). \tag{8.20}$$

Indeed if $\lambda \in \rho(A)$ then by part 2 $(\lambda I - A)^{-1}$ maps X^+ into X^+. Thus $(\lambda I_{X^+} - A^+)(\lambda I - A)^{-1}|_{X^+}$ is well defined and

$$(\lambda I_{X^+} - A^+)(\lambda I - A)^{-1}|_{X^+} = (\lambda I - A)(\lambda I - A)^{-1}|_{X^+} = I_{X^+}.$$

On the other hand, on X^+ we have that

$$\begin{aligned}(\lambda I - A)^{-1}|_{X^+}(\lambda I_{X^+} - A^+) &= (\lambda I - A)^{-1}|_{X^+}(\lambda I - A)|_{X^+} \\ &= (\lambda I - A)^{-1}(\lambda I - A)|_{X^+} \qquad \text{(by part 2)} \\ &= I_{X^+}.\end{aligned}$$

So $\rho(A) \subset \rho(A^+)$ and $(\lambda I - A)^{-1}|_{X^+} = (\lambda I_{X^+} - A^+)^{-1}$. Similarly, we can show that $\rho(A) \subset \rho(A^-)$ and $(\lambda I - A)^{-1}|_{X^-} = (\lambda I_{X^-} - A^-)^{-1}$. Thus we deduce that $\rho(A) \subset \rho(A^+) \cap \rho(A^-)$.

If $s \in \rho(A^+) \cap \rho(A^-)$, then by using the following decomposition for $x \in D(A)$,

$$(sI - A)x = (sI - A^+)P_\Gamma x + (sI - A^-)(I - P_\Gamma)x, \tag{8.21}$$

it is easy to see that $(sI - A^+)^{-1}P_\Gamma + (sI - A^-)^{-1}(I - P_\Gamma)$ is the inverse of $(sI - A)$. Thus (8.20) is proved.

By definition, we have that $\sigma(A) = \sigma^+ \cup \sigma^-$. Furthermore, we showed that $\sigma^+ \subset \rho(A^-)$ and $\sigma^- \subset \rho(A^+)$. Combining this with (8.20) we find $\sigma(A^+) = \sigma^+$ and $\sigma(A^-) = \sigma^-$.

Part 5. We may write $\sigma^+ = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. By standard complex integration theory we have that

$$\begin{aligned}P_\Gamma x &= \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x d\lambda \\ &= \sum_{n=1}^{N} \frac{1}{2\pi i}\int_{\Gamma_n} (\lambda I - A)^{-1} x d\lambda = \sum_{n=1}^{N} P_{\Gamma_n} x,\end{aligned}$$

where Γ_n is a rectifiable, closed, simple curve enclosing only λ_n. Thus it suffices to prove the assertion for the case that $\sigma^+ = \{\lambda_n\}$, where λ_n is an eigenvalue with finite order $\nu(n)$. We do this for the generic case $\nu(n) = 1$; the general case can be found in [10]. Let Γ denote the rectifiable, simple, closed curve that encloses counterclockwise only one point λ_n in the spectrum of A.

First we prove that $\operatorname{ran} P_\Gamma \subset \ker(\lambda_n I - A)$. For $x \in X$ and $s \in \rho(A)$ we have

$$(\lambda_n - s)P_\Gamma x = \frac{1}{2\pi i}\int_\Gamma (\lambda_n - s)(\lambda I - A)^{-1} x d\lambda. \tag{8.22}$$

Multiplying (8.15) from the left by $(sI - A)$ and adding (8.22) yields

$$(\lambda_n I - A)P_\Gamma x = \frac{1}{2\pi i}\int_\Gamma (\lambda_n - \lambda)(\lambda I - A)^{-1} x d\lambda.$$

This last expression is zero, since $(\lambda_n - \cdot)(\cdot I - A)^{-1}x$ is holomorphic inside Γ. This proves $\operatorname{ran} P_\Gamma \subset \ker(\lambda_n I - A)$; to prove the other inclusion note that $(\lambda_n I - A)x_0 = 0$ implies that $(\lambda I - A)^{-1}x_0 = \frac{1}{\lambda - \lambda_n}x_0$. This implies

$$P_\Gamma x_0 = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x_0 d\lambda = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda - \lambda_n} x_0 d\lambda = x_0.$$

Part 6. In part 5 we showed that P_Γ maps onto the span of ϕ_n. Hence $P_\Gamma z = h(z)\phi_n$, where h is a function from X to $\mathbb{C}$. Since P_Γ is a bounded linear operator, it follows that h is an element of $\mathcal{L}(X, \mathbb{C})$. From the Riesz Representation Theorem it follows that $h(z) = \langle z, \psi_n \rangle$ for some $\psi_n \in X$. Consider now for $x \in D(A)$,

$$\begin{aligned}\langle x, A^*\psi_n\rangle\phi_n &= \langle Ax, \psi_n\rangle\phi_n = P_\Gamma Ax \\ &= AP_\Gamma x = A\langle x, \psi_n\rangle\phi_n = \langle x, \overline{\lambda_n}\psi_n\rangle\phi_n.\end{aligned}$$

Since this holds for every $x \in D(A)$, we conclude that $A^*\psi_n = \overline{\lambda_n}\psi_n$. Furthermore, using the fact that $P_\Gamma\phi_n = \phi_n$ it follows easily that $\langle \phi_n, \psi_n \rangle = 1$. □

Theorem 8.2.5. *Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on X. Assume that the spectrum of A is the union of two parts, σ^+ and σ^- as in Theorem 8.2.4. Then X^+ and X^- are $T(t)$-invariant and $(T^+(t))_{t\geq 0}$, $(T^-(t))_{t\geq 0}$ with $T^+(t) = T(t)|_{X^+}$, $T^-(t) = T(t)|_{X^-}$ define C_0-semigroups on X^+ and X^-, respectively. The infinitesimal generator of $(T^+(t))_{t\geq 0}$ is A^+, whereas A^- is the infinitesimal generator of $(T^-(t))_{t\geq 0}$.*

Proof. Since $(\lambda I - A)^{-1}T(t) = T(t)(\lambda I - A)^{-1}$ for all $\lambda \in \rho(A)$ and $t \geq 0$, it follows that P_Γ commutes with $T(t)$. Then it is easily proved that X^+ and X^- are $T(t)$-invariant. Consequently, $(T^+(t))_{t\geq 0}$ and $(T^-(t))_{t\geq 0}$ with $T^+(t) = T(t)|_{X^+}$, $T^-(t) = T(t)|_{X^-}$ are C_0-semigroups on X^+ and X^-, respectively.

We shall only prove that A^- is the infinitesimal generator of $(T^-(t))_{t\geq 0}$, as the proof for $(T^+(t))_{t\geq 0}$ is very similar. Suppose that $\lim\limits_{t\to 0^+} \frac{T^-(t)x - x}{t}$ exists for $x \in X^-$. Since $T^-(t)x = T(t)x$ we conclude that $x \in D(A)$ and hence x is an element of $D(A^-)$. By definition, the limit equals $Ax = A^-x$. On the other hand, if $x \in D(A^-)$, then $x \in D(A)$ and so the limit exists and equals A^-x. Combining these results the infinitesimal generator of $(T^-(t))_{t\geq 0}$ is A^-. □

We combine the previous theorem with Theorem 8.1.4 to find a sufficient condition guaranteeing that $(T^-(t))_{t\geq 0}$ is exponentially stable.

Lemma 8.2.6. *Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on X. Assume that the spectrum of A is the union of two parts, σ^+ and σ^-, such that $\sigma^+ \subset \overline{\mathbb{C}_0^+} = \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}$, $\sigma^- \subset \mathbb{C}_0^- = \{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}$, and that a rectifiable, closed, simple curve Γ can be drawn so as to enclose an open set containing σ^+ in its interior and σ^- in its exterior. Assume further that*

$$\sup_{s\in\mathbb{C}_0^+,\, s \text{ outside } \Gamma} \|(sI - A)^{-1}\| < \infty. \tag{8.23}$$

Then the semigroup $(T^-(t))_{t\geq 0} := (T(t)|_{X^-})_{t\geq 0}$ is exponentially stable, where X^- is the subspace of X associated to σ^-, see (8.12).

Proof. By Theorem 8.2.5 the infinitesimal generator of $(T(t)|_{X^-})_{t\geq 0}$ equals A^-. The spectrum of A^- lies outside Γ, see Theorem 8.2.4.4. Since the interior of Γ is a bounded set, and since the resolvent is holomorphic on the resolvent set, we find that

$$\sup_{s \text{ inside } \Gamma} \|(sI - A^-)^{-1}\| < \infty. \tag{8.24}$$

Furthermore, for $x^- \in X^-$ and $s \in \mathbb{C}_0^+$ outside Γ, we have that

$$(sI - A^-)^{-1}x^- = (sI - A)^{-1}x^-.$$

Combining this with (8.23) and (8.24) gives that the resolvent operator of A^- is uniformly bounded on $\mathbb{C}_0^+$, and by Theorem 8.1.4 we conclude that $(T^-(t))_{t\geq 0}$ is exponentially stable. □

8.3 Exercises

8.1. In this exercise we introduce a different concept of stability, and show that it is weaker than exponential stability.

Definition 8.3.1. The C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X is *strongly stable* if for every $x \in X$, $T(t)x$ converges to zero as t tends to ∞.

Let X be the Hilbert space $L^2(0,\infty)$ and let the operators $T(t) : X \to X$, $t \geq 0$, be defined by

$$(T(t)f)(\zeta) := f(t + \zeta).$$

a) Show that $(T(t))_{t\geq 0}$ is a C_0-semigroup on X.

b) Prove that $(T(t))_{t\geq 0}$ is strongly stable, but not exponentially stable.

8.2. A natural question is whether the following condition is sufficient for strong or exponential stability,

$$\langle Ax, x\rangle + \langle x, Ax\rangle < 0, \qquad \text{for all } x \in D(A) \setminus \{0\}. \tag{8.25}$$

In this exercise we show that this does not hold in general.

Let X be the Hilbert space $L^2(0,\infty)$ equipped with the inner product

$$\langle f, g\rangle := \int_0^\infty f(\zeta)\overline{g(\zeta)}(e^{-\zeta} + 1)d\zeta,$$

and let the operators $T(t) : X \to X$, $t \geq 0$, be defined by

$$(T(t)f)(\zeta) := \begin{cases} f(\zeta - t), & \zeta > t, \\ 0, & 0 \leq \zeta < t. \end{cases}$$

a) Show that $(T(t))_{t\geq 0}$ is a C_0-semigroup on X.

b) Prove that $(T(t))_{t\geq 0}$ is not strongly stable.

c) Prove $\|T(t_2)x\| < \|T(t_1)x\|$ for all $x \in X \setminus \{0\}$ and $t_2 > t_1 \geq 0$.

d) Show that the infinitesimal generator of $(T(t))_{t\geq 0}$ is given by

$$Af = -\frac{df}{d\zeta}$$

with domain

$$D(A) = \{f \in X \mid f \text{ is absolutely continuous}, \frac{df}{d\zeta} \in X, \text{ and } f(0) = 0\}.$$

Hint: See Exercise 6.3.b.

e) Use the previous item to prove that (8.25) holds.

8.3. Let Q be a bounded, self-adjoint operator on the Hilbert space X which satisfies $Q \geq mI$, for some $m > 0$. Let A be the infinitesimal generator of a strongly continuous semigroup on X which satisfies the Lyapunov inequality

$$\langle Ax, Qx\rangle + \langle Qx, Ax\rangle \leq 0, \qquad x \in D(A).$$

Show that there exists an equivalent norm on X such that A generates a contraction semigroup with respect to this new norm.

8.4. Let A be a bounded operator on the Hilbert space X, and let V be a closed linear subspace of X. Show that V is $T(t)$-invariant if and only if V is A-invariant.

8.5. Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ and let V be a one-dimensional linear subspace of X. Show that V is $T(t)$-invariant if and only if $V = \text{span}\{\phi\}$ with ϕ an eigenvector of A.

Hint: You may use the following fact: If a continuous scalar function f satisfies $f(t+s) = f(t)f(s)$ for $t, s \in (0, \infty)$, then $f(t) = e^{\lambda t}$ for some λ.

8.4 Notes and references

Exponential stability is one of the most studied properties of semigroups, and therefore the results in the first section can be found in many books, we refer to chapter 5 of Curtain and Zwart [10] or chapter V of Engel and Nagel [15]. The characterization as presented in Theorem 8.1.4 originates from Huang [26]. The relation between $T(t)$- and A-invariance is studied in [45], [63], see also [10]. Spectral projections play an important role in many subfields of functional analysis, and thus the results as formulated in Theorem 8.2.4 can be found at many places, see [10, Lemma 2.5.7] or [18, Theorem XV.2.1].

Chapter 9

Stability of Port-Hamiltonian Systems

In this chapter we return to the class of port-Hamiltonian partial differential equations which we introduced in Chapter 7. If a port-Hamiltonian system possesses n (linearly independent) boundary conditions and if the energy is non-increasing, then the associated abstract differential operator generates a contraction semigroup on the energy space. This chapter is devoted to exponential stability of port-Hamiltonian systems. Exercise 8.2 implies that the condition

$$\langle Ax, x\rangle + \langle x, Ax\rangle < 0, \quad x \in D(A), x \neq 0, \tag{9.1}$$

is in general not sufficient for strong stability of the semigroup generated by A. However, in this chapter we show that if a weaker but more structured condition than (9.1) holds with respect to the energy inner product for a port-Hamiltonian system, then the port-Hamiltonian system is even exponentially stable, see Theorem 9.1.3.

9.1 Exponential stability of port-Hamiltonian systems

We start with a repetition of the definition of homogeneous port-Hamiltonian systems as introduced in Section 7.1. Moreover, we equip the port-Hamiltonian system with boundary conditions, that is we study differential equations of the form

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(\zeta, t)\right) + P_0\left(\mathcal{H}(\zeta)x(\zeta, t)\right) \tag{9.2}$$

with the boundary condition

$$W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0, \tag{9.3}$$

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_9,

where

$$\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix}. \tag{9.4}$$

We assume that the following conditions hold, see Definition 7.1.2 and Theorem 7.2.4.

Assumption 9.1.1.

- $P_1 \in \mathbb{K}^{n\times n}$ is invertible and self-adjoint;
- $P_0 \in \mathbb{K}^{n\times n}$ is skew-adjoint;
- $\mathcal{H} \in C^1([a,b];\mathbb{K}^{n\times n})$, $\mathcal{H}(\zeta)$ is self-adjoint for all $\zeta \in [a,b]$ and $mI \leq \mathcal{H}(\zeta) \leq MI$ for all $\zeta \in [a,b]$ and some $M, m > 0$ independent of ζ;
- $W_B \in \mathbb{K}^{n\times 2n}$ has full rank;
- $W_B \Sigma W_B^* \geq 0$, where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$.

We note that Assumption 9.1.1 is slightly stronger than the assumptions used in Chapter 7 as we did not assume that $\mathcal{H}$ is continuous differentiable in Chapter 7. However, we would like to remark that our main theorem (Theorem 9.1.3) also holds if P_0 satisfies $P_0 + P_0^* \leq 0$. Throughout this section we assume that Assumption 9.1.1 is satisfied. Theorem 7.2.4 implies that the operator A given by

$$Ax := P_1 \frac{d}{d\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x) \tag{9.5}$$

with domain

$$D(A) = \{x \in L^2([a,b];\mathbb{K}^n) \mid \mathcal{H}x \in H^1([a,b];\mathbb{K}^n),\ W_B \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] = 0\} \tag{9.6}$$

generates a contraction semigroup on the state space

$$X = L^2([a,b];\mathbb{K}^n) \tag{9.7}$$

equipped with the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^* \mathcal{H}(\zeta) f(\zeta) d\zeta. \tag{9.8}$$

The associated norm is denoted by $\|\cdot\|_X$.

In the following lemma, we show that the norm/energy of a state trajectory can be bounded by the energy at one of the boundaries. This result is essential for the proof of exponential stability of port-Hamiltonian systems. The proof is based on an idea of Cox and Zuazua in [8].

Lemma 9.1.2. *Let A be defined by* (9.5) *and* (9.6)*, and let $(T(t))_{t\geq 0}$ be the contraction semigroup generated by A. Then there exists constants $\tau > 0$ and $c > 0$ such that for every $x_0 \in D(A)$ the state trajectory $x(t) := T(t)x_0$ satisfies*

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|\mathcal{H}(b)x(b,t)\|^2 \, dt \qquad \textit{and} \tag{9.9}$$

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau \|\mathcal{H}(a)x(a,t)\|^2 \, dt. \tag{9.10}$$

Proof. Let $x_0 \in D(A)$ be arbitrary and let $x(t) := T(t)x_0$, $t \geq 0$. $x_0 \in D(A)$ implies that $x(t) \in D(A)$ for all t. Furthermore, x is the classical solution of (9.2) with $x(0) = x_0$. In the following we write $x(\zeta, t)$ instead of $(x(t))(\zeta)$.

For this trajectory, we define the function $F : [a, b] \to \mathbb{R}$ by

$$F(\zeta) = \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\,dt, \quad \zeta \in [a,b], \tag{9.11}$$

where we assume that $\gamma > 0$ and $\tau > 2\gamma(b-a)$. The second condition implies in particular $\tau - \gamma(b-\zeta) > \gamma(b-\zeta)$, that is, we are not integrating over a negative time interval. Differentiating the function F gives

$$\begin{aligned}
\frac{dF}{d\zeta}(\zeta) = &\int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*(\zeta,t)\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t))\,dt \\
&+ \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} \left(\frac{\partial}{\partial\zeta}x(\zeta,t)\right)^* \mathcal{H}(\zeta)x(\zeta,t)\,dt \\
&+ \gamma x^*(\zeta,\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\gamma(b-\zeta)) \\
&+ \gamma x^*(\zeta,\tau-\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\tau-\gamma(b-\zeta)).
\end{aligned}$$

Since P_1 is invertible and since x satisfies (9.2), we obtain (for simplicity we omit the dependence on ζ and t)

$$\begin{aligned}
\frac{dF}{d\zeta}(\zeta) &= \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* P_1^{-1}\left(\frac{\partial x}{\partial t} - P_0\mathcal{H}x\right) dt \\
&\quad + \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} \left(P_1^{-1}\frac{\partial x}{\partial t} - \frac{d\mathcal{H}}{d\zeta}x - P_1^{-1}P_0\mathcal{H}x\right)^* x\,dt \\
&\quad + \gamma x^*(\zeta,\tau-\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\tau-\gamma(b-\zeta)) \\
&\quad + \gamma x^*(\zeta,\gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta,\gamma(b-\zeta)) \\
&= \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* P_1^{-1}\frac{\partial x}{\partial t} + \frac{\partial x}{\partial t}^* P_1^{-1}x\,dt - \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*\frac{d\mathcal{H}}{d\zeta}\,x\,dt \\
&\quad - \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*\left(\mathcal{H}P_0^*P_1^{-1} + P_1^{-1}P_0\mathcal{H}\right)x\,dt
\end{aligned}$$

$$+ \gamma x^*(\zeta, \tau - \gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta, \tau - \gamma(b-\zeta))$$
$$+ \gamma x^*(\zeta, \gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta, \gamma(b-\zeta))$$

where we have used that $P_1^* = P_1$, $\mathcal{H}^* = \mathcal{H}$. The first integral can be calculated, and so we find

$$\begin{aligned}
\frac{dF}{d\zeta}(\zeta) = {} & x^*(\zeta,t)P_1^{-1}x(\zeta,t)\Big|_{t=\gamma(b-\zeta)}^{t=\tau-\gamma(b-\zeta)} - \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* \frac{d\mathcal{H}}{d\zeta} x\,dt \\
& - \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* \left(\mathcal{H}P_0^*P_1^{-1} + P_1^{-1}P_0\mathcal{H}\right) x\,dt \\
& + \gamma x^*(\zeta, \tau - \gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta, \tau - \gamma(b-\zeta)) \\
& + \gamma x^*(\zeta, \gamma(b-\zeta))\mathcal{H}(\zeta)x(\zeta, \gamma(b-\zeta)) \\
= {} & - \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* \left(\mathcal{H}P_0^*P_1^{-1} + P_1^{-1}P_0\mathcal{H} + \frac{d\mathcal{H}}{d\zeta}\right) x\,dt \\
& + x^*(\zeta, \tau - \gamma(b-\zeta))\left(P_1^{-1} + \gamma\mathcal{H}(\zeta)\right)x(\zeta, \tau - \gamma(b-\zeta)) \\
& + x^*(\zeta, \gamma(b-\zeta))\left(-P_1^{-1} + \gamma\mathcal{H}(\zeta)\right)x(\zeta, \gamma(b-\zeta)).
\end{aligned}$$

We now choose γ large enough, such that $P_1^{-1} + \gamma\mathcal{H}$ and $-P_1^{-1} + \gamma\mathcal{H}$ are coercive (positive definite). Note that $\tau > 2\gamma(b-a)$, and thus a large constant γ implies that τ must be large as well. Using the coercivity of $P_1^{-1} + \gamma\mathcal{H}$ and $-P_1^{-1} + \gamma\mathcal{H}$, we find that

$$\frac{dF}{d\zeta}(\zeta) \geq - \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^* \left(\mathcal{H}P_0^*P_1^{-1} + P_1^{-1}P_0\mathcal{H} + \frac{d\mathcal{H}}{d\zeta}\right) x\,dt.$$

Since P_1 and P_0 are constant matrices and, by assumption, $\frac{d\mathcal{H}}{d\zeta}(\zeta)$ is bounded, there exists a constant $\kappa > 0$ such that for all $\zeta \in [a,b]$ we have

$$\mathcal{H}(\zeta)P_0^*P_1^{-1} + P_1^{-1}P_0\mathcal{H}(\zeta) + \frac{d\mathcal{H}}{d\zeta} \leq \kappa\mathcal{H}(\zeta).$$

Thus, for all $\zeta \in [a,b]$ we obtain

$$\frac{dF}{d\zeta}(\zeta) \geq -\kappa \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\,dt = -\kappa\, F(\zeta), \tag{9.12}$$

where we used (9.11). For simplicity, we denote the derivative of F by F'. Inequality (9.12) implies, for all $\zeta_1 \in [a,b]$,

$$\int_{\zeta_1}^{b} \frac{F'(\zeta)}{F(\zeta)}\,d\zeta \geq -\kappa \int_{\zeta_1}^{b} 1\,d\zeta, \tag{9.13}$$

or equivalently

$$\ln\big(F(b)\big) - \ln\big(F(\zeta_1)\big) \geq -\kappa\,(b - \zeta_1). \tag{9.14}$$

Thus we obtain

$$F(b) \geq F(\zeta_1)\, e^{-\kappa\,(b-\zeta_1)} \geq F(\zeta_1)\, e^{-\kappa\,(b-a)} \quad \text{for} \quad \zeta_1 \in [a,b]. \tag{9.15}$$

On the other hand, since $\|x(t_2)\|_X \leq \|x(t_1)\|_X$ for any $t_2 \geq t_1$ (by the contraction property of the semigroup), we deduce that

$$\begin{aligned}\int_{\gamma(b-a)}^{\tau-\gamma(b-a)} \|x(t)\|_X^2\, dt &\geq \|x(\tau-\gamma(b-a))\|_X^2 \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} 1\, dt \\ &= (\tau - 2\gamma(b-a))\|x(\tau-\gamma(b-a))\|_X^2.\end{aligned}$$

Using the definition of F and $\|x(t)\|_X^2$, see (9.11) and (9.8), together with the equation above, the estimate (9.15), and the coercivity of $\mathcal{H}$ we obtain

$$\begin{aligned}2(\tau-2\gamma(b-a))\|x(\tau)\|_X^2 &\leq 2(\tau-2\gamma(b-a))\|x(\tau-\gamma(b-a))\|_X^2 \\ &\leq 2\int_{\gamma(b-a)}^{\tau-\gamma(b-a)} \|x(t)\|_X^2\, dt \\ &= \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} \int_a^b x^*(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, d\zeta\, dt \\ &= \int_a^b \int_{\gamma(b-a)}^{\tau-\gamma(b-a)} x^*(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, dt\, d\zeta \\ &\qquad \text{(by Fubini's theorem)} \\ &\leq \int_a^b \int_{\gamma(b-\zeta)}^{\tau-\gamma(b-\zeta)} x^*(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, dt\, d\zeta \\ &\qquad \text{(since the integration interval increases)} \\ &= \int_a^b F(\zeta)\, d\zeta \leq (b-a)F(b)\, e^{\kappa\,(b-a)} \\ &= (b-a)\, e^{\kappa\,(b-a)} \int_0^\tau x^*(b,t)\mathcal{H}(b)x(b,t)\, dt \\ &\leq m^{-1}(b-a)\, e^{\kappa\,(b-a)} \int_0^\tau \|\mathcal{H}(b)x(b,t)\|^2\, dt.\end{aligned}$$

Hence for our choice of τ we have that

$$\|x(\tau)\|_X^2 \leq c\int_0^\tau \|\mathcal{H}(b)x(b,t)\|^2\, dt, \tag{9.16}$$

where $c = \frac{(b-a)\, e^{\kappa\,(b-a)}}{2(\tau-2\gamma(b-a))m}$. This proves estimate (9.9).

The second estimate follows in a similar manner by replacing the function F in the calculations above by

$$\tilde{F}(\zeta) = \int_{\gamma(\zeta-a)}^{\tau-\gamma(\zeta-a)} x^*(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)\, dt. \qquad \square$$

Using this technical lemma the proof of exponential stability is easy.

Theorem 9.1.3. *Let A be defined by* (9.5) *and* (9.6)*. If for some positive constant k one of the following conditions is satisfied for all $x_0 \in D(A)$,*

$$\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X \leq -k\|\mathcal{H}(b)x_0(b)\|^2 \tag{9.17}$$

$$\langle Ax_0, x_0\rangle_X + \langle x_0, Ax_0\rangle_X \leq -k\|\mathcal{H}(a)x_0(a)\|^2, \tag{9.18}$$

then A generates an exponentially stable C_0-semigroup.

Proof. Without loss of generality we assume that the first inequality (9.17) holds. By Lemma 9.1.2 there exist positive constants τ and c such that (9.9) holds. Let $x_0 \in D(A)$ and by $(T(t))_{t\geq 0}$ we denote the C_0-semigroup generated by A. We define $x(t) := T(t)x_0$, $t \geq 0$. $x_0 \in D(A)$ implies $x(t) \in D(A)$ and $\dot{x}(t) = Ax(t)$, $t \geq 0$. Using this differential equation, we obtain

$$\frac{d\|x(t)\|_X^2}{dt} = \frac{d\langle x(t), x(t)\rangle_X}{dt} = \langle Ax(t), x(t)\rangle_X + \langle x(t) Ax(t)\rangle_X. \tag{9.19}$$

Equations (9.17) and (9.19) now imply that

$$\begin{aligned}\|x(\tau)\|_X^2 - \|x(0)\|_X^2 &= \int_0^\tau \frac{d\|x(t)\|_X^2}{dt}\,dt \\ &\leq -k\int_0^\tau \|\mathcal{H}(b)x(b,t)\|^2 dt.\end{aligned}$$

Combining this with (9.9), we find that

$$\|x(\tau)\|_X^2 - \|x(0)\|_X^2 \leq \frac{-k}{c}\|x(\tau)\|_X^2.$$

Thus $\|x(\tau)\|_X^2 \leq \frac{c}{c+k}\|x(0)\|_X^2$, which implies that the semigroup $(T(t))_{t\geq 0}$ generated by A satisfies $\|T(\tau)\| < 1$, and therefore $(T(t))_{t\geq 0}$ is exponentially stable by Theorem 5.1.5. □

Estimate (9.17) provides a simple test for exponential stability of port-Hamiltonian systems. We note that Lemma 7.2.1 and (7.23) imply

$$\begin{aligned}\langle Ax, x\rangle_X + \langle x, Ax\rangle_X &= \frac{1}{2}\left((\mathcal{H}(b)x(b))^* P_1 \mathcal{H}(b)x(b) - (\mathcal{H}(a)x(a))^* P_1 \mathcal{H}(a)x(a)\right) \\ &= \frac{1}{2}\left(f_\partial^* e_\partial + e_\partial^* f_\partial\right).\end{aligned} \tag{9.20}$$

This equality can be used to establish (9.17) or (9.18). An even simpler sufficient condition is provided by the following lemma.

Lemma 9.1.4. *Consider the port-Hamiltonian system* (9.2)–(9.3)*. If $W_B \Sigma W_B^* > 0$, then the port-Hamiltonian system is exponentially stable.*

Proof. Lemma 7.3.1 implies that W_B can be written as $W_B = S\begin{bmatrix} I+V & I-V \end{bmatrix}$, where $S, V \in \mathbb{K}^{n\times n}$, S is invertible and $VV^* < I$.

We define $W_C = \begin{bmatrix} I+V^* & -I+V^* \end{bmatrix}$. It is easy to see that with this choice, W_C is a $n\times 2n$ matrix with rank n. Next we prove that the matrix $\left[\begin{smallmatrix} I+V & I-V \\ I+V^* & -I+V^* \end{smallmatrix}\right]$ is invertible. Assuming $\left[\begin{smallmatrix} I+V & I-V \\ I+V^* & -I+V^* \end{smallmatrix}\right]$ is not invertible, there exists a vector $\begin{bmatrix} x & y \end{bmatrix} \neq 0$ such that $\begin{bmatrix} x & y \end{bmatrix}\left[\begin{smallmatrix} I+V & I-V \\ I+V^* & -I+V^* \end{smallmatrix}\right] = 0$, or equivalently $(I+V)x + (I+V^*)y = 0$ and $(I-V)x + (-I+V^*)y = 0$. These equations together with $VV^* < I$ imply $x = y = 0$, and thus the matrix $\left[\begin{smallmatrix} I+V & I-V \\ I+V^* & -I+V^* \end{smallmatrix}\right]$ is invertible. This implies the invertibility of the matrix $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$.

Let $x \in D(A)$ be arbitrary. The definition of the domain of A implies that $\left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] \in \ker W_B$. Lemma 7.3.2 implies that

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I-V \\ -I-V \end{bmatrix} \ell \tag{9.21}$$

for some $\ell \in \mathbb{K}^n$. Thus with (9.20) we find

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \ell^*(-I+V^*V)\ell. \tag{9.22}$$

Furthermore, we have the equality

$$y := W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I+V^*, -I+V^* \end{bmatrix} \begin{bmatrix} I-V \\ -I-V \end{bmatrix} \ell = 2(I-V^*V)\ell. \tag{9.23}$$

Combining equations (9.22) and (9.23) we obtain

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \frac{1}{4} y^*[-I+V^*V]^{-1} y \leq -m_1\|y\|^2 \tag{9.24}$$

for some $m_1 > 0$. Here we have used that $VV^* < I$, or equivalently $V^*V - I < 0$.

Using (9.4), the boundary condition, and the definition of y the relation between y and x is given by

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} W_B \\ W_C \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} =: W \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix}.$$

Since P_1 and $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ are invertible, it follows that the matrix W is invertible and, in particular, $\|W w\|^2 \geq m_2\|w\|^2$ for every $w \in \mathbb{K}^n$ and some $m_2 > 0$. Taking norms on both sides yields

$$\|y\|^2 = \left\| W \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} \right\|^2 \geq m_2 \left\| \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} \right\|^2 \geq m_2\|\mathcal{H}(b)x(b)\|^2. \tag{9.25}$$

Combining the estimates (9.24) and (9.25) we obtain

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X \leq -m_1\|y\|^2 \leq -m_1 m_2 \|(\mathcal{H}x)(b)\|^2.$$

Thus (9.17) holds, and therefore Theorem 9.1.3 implies the exponential stability of the port-Hamiltonian system. □

Unfortunately, the sufficient condition of the previous lemma often cannot be applied, as the condition implies that the system possesses as many dampers as boundary controls. In practice fewer dampers are necessary as it is shown in the example of the following section.

9.2 An example

In this section we show how to apply the results of the previous section to the vibrating string.

Example 9.2.1. Consider the *vibrating string* on the spatial interval $[a, b]$. In Example 7.1.1 we saw that the model is given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta, t) \right), \tag{9.26}$$

where $\zeta \in [a, b]$ is the spatial variable, $w(\zeta, t)$ is the vertical position of the string at position ζ and time t, T is the Young's modulus of the string, and ρ is the mass density. This system has the energy/Hamiltonian

$$E(t) = \frac{1}{2} \int_a^b \rho(\zeta) \left(\frac{\partial w}{\partial t}(\zeta, t) \right)^2 + T(\zeta) \left(\frac{\partial w}{\partial \zeta}(\zeta, t) \right)^2 d\zeta. \tag{9.27}$$

As depicted in Figure 9.1, the string is fixed at the left-hand side and at

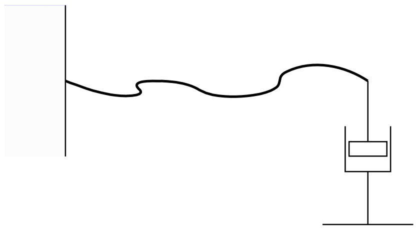

Figure 9.1: The vibrating string with a damper

the right-hand side we attach a damper, that is, the force at this tip equals a (negative) constant times the velocity of the tip, i.e,

$$T(b) \frac{\partial w}{\partial \zeta}(b, t) = -k \frac{\partial w}{\partial t}(b, t), \qquad k \geq 0. \tag{9.28}$$

For this example the boundary effort and boundary flow are given by, see Example 7.2.5,

$$f_\partial = \frac{1}{\sqrt{2}} \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \end{bmatrix}, \; e_\partial = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a) \end{bmatrix}. \tag{9.29}$$

The boundary conditions $\frac{\partial w}{\partial t}(a) = 0$ and (9.28) can be equivalently written as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b) + k\frac{\partial w}{\partial t}(b) \\ \frac{\partial w}{\partial t}(a) \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & k & k & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \tag{9.30}$$

with $W_B = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & k & k & 1 \\ 0 & -1 & 1 & 0 \end{smallmatrix}\right]$. W_B is a 2×4-matrix with rank 2, and

$$W_B \Sigma W_B^* = \begin{bmatrix} 2k & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the matrix $W_B \Sigma W_B^*$ is non-negative, the infinitesimal generator associated to this p.d.e., generates a contractions semigroup. However, $W_B \Sigma W_B^*$ is not positive definite, and so we cannot used Lemma 9.1.4 to conclude that the corresponding semigroup is exponentially stable. We will show that (9.17) holds.

As state variables we have chosen $x = \left[\begin{smallmatrix} \rho \frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{smallmatrix}\right]$ and $\mathcal{H}$ is given by $\mathcal{H}(\zeta) = \left[\begin{smallmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{smallmatrix}\right]$. By equation (9.20) we have that

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X = \frac{\partial w}{\partial t}(b)T(b)\frac{\partial w}{\partial \zeta}(b) - \frac{\partial w}{\partial t}(a)T(a)\frac{\partial w}{\partial \zeta}(a)$$
$$= -k\left(\frac{\partial w}{\partial t}(b)\right)^2, \tag{9.31}$$

where we have used the boundary conditions. Moreover, we calculate

$$\|\mathcal{H}(b)x(b)\|^2 = \left(\frac{\partial w}{\partial t}(b)\right)^2 + \left(T(b)\frac{\partial w}{\partial \zeta}(b)\right)^2 = (k^2+1)\left(\frac{\partial w}{\partial t}(b)\right)^2. \tag{9.32}$$

Combining the two previous equations, we find that

$$\langle Ax, x\rangle_X + \langle x, Ax\rangle_X \leq -\frac{k}{1+k^2}\|\mathcal{H}(b)x(b)\|^2. \tag{9.33}$$

Hence using Theorem 9.1.3 we may conclude that attaching a damper at one end of the transmission line stabilizes the system exponentially.

We remark that if the damper is not connected, i.e., $k = 0$, then the corresponding semigroup is a unitary group, and the solution cannot be exponentially stable.

9.3 Exercises

9.1. Consider the *transmission line* on the spatial interval $[a, b]$ as discussed in Exercise 7.1

$$\frac{\partial Q}{\partial t}(\zeta, t) = -\frac{\partial}{\partial \zeta}\frac{\phi(\zeta, t)}{L(\zeta)}, \qquad (9.34)$$
$$\frac{\partial \phi}{\partial t}(\zeta, t) = -\frac{\partial}{\partial \zeta}\frac{Q(\zeta, t)}{C(\zeta)}.$$

Here $Q(\zeta, t)$ is the charge at position $\zeta \in [a, b]$ and time $t > 0$, and $\phi(\zeta, t)$ is the (magnetic) flux at position ζ and time t. C is the (distributed) capacity and L is the (distributed) inductance.

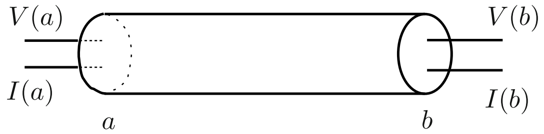

Figure 9.2: Transmission line

The voltage and current are given by $V = Q/C$ and $I = \phi/L$, respectively.

We set the voltage at $\zeta = a$ to zero, and put a resistor at the other end. This implies that we have the p.d.e. (9.34) with boundary conditions

$$V(a, t) = 0, \qquad V(b, t) = RI(b, t), \qquad (9.35)$$

with $R > 0$.

(a) Show that the differential operator associated to the p.d.e. (9.34) with boundary conditions (9.35) generates a contraction semigroup on the energy space, see (7.53).

(b) Prove that the semigroup as defined in the previous item is exponentially stable.

9.2. Consider a flexible beam modeled by the Timoshenko beam equations, see Example 7.1.4. We assume that the beam is clamped at the left-hand side, i.e., at $\zeta = a$, and at the right-hand side we apply a damping force proportional to the velocity. Thus the boundary conditions are

$$\frac{\partial w}{\partial t}(a, t) = 0, \qquad \frac{\partial \phi}{\partial t}(a, t) = 0, \qquad EI(b)\frac{\partial \phi}{\partial \zeta}(b, t) = -\alpha_1 \frac{\partial \phi}{\partial t}(b, t) \qquad (9.36)$$

and

$$K(b)\left(\frac{\partial w}{\partial \zeta}(b, t) - \phi(b, t)\right) = -\alpha_2 \frac{\partial w}{\partial t}(b, t). \qquad (9.37)$$

(a) Assume that $\alpha_1, \alpha_2 \in [0, \infty)$. Show that under these conditions the Timoshenko beam associated with boundary conditions (9.36) and (9.37) generates a contraction semigroup on its energy space.

(b) Prove that the semigroup as defined in the previous item is exponentially stable if α_1 and α_2 are positive.

(c) Show that the semigroup as defined in the first item is not exponentially stable if $\alpha_1 = 0$ and $\alpha_2 = 0$.

9.3. Consider coupled vibrating strings as given in the figure below. We assume

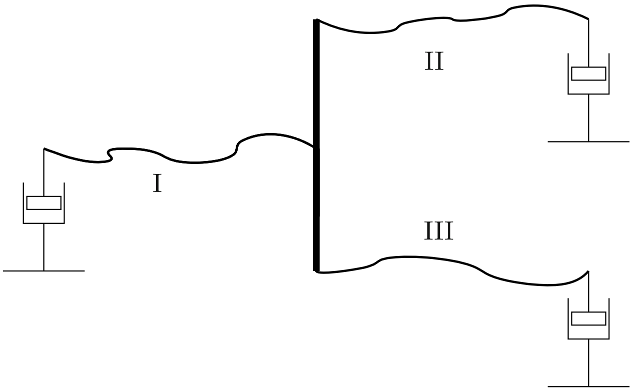

Figure 9.3: Coupled vibrating strings with dampers

that the length of all strings are equal. The model for every vibrating string is given by (9.26) with physical parameters, $\rho_{\mathrm{I}}, T_{\mathrm{I}}, \rho_{\mathrm{II}}$, etc. Furthermore, we assume that the three strings are connected via a (mass-less) bar, as shown in Figure 9.3. This bar can only move in the vertical direction. This implies that the velocity of string I, $\frac{\partial w_I}{\partial t}$, at its right-hand side equals those of the other two strings at their left-hand side. Furthermore, the force of string I at its right-end side equals the sum of the forces of the other two at their left-hand side. We assume that for all three waves, the connecting point is at coordinate $\zeta = a$. Thus the balance of forces in the middle is given by

$$-T_{\mathrm{I}}(a)\frac{\partial w_{\mathrm{I}}}{\partial \zeta}(a) = T_{\mathrm{II}}(a)\frac{\partial w_{\mathrm{II}}}{\partial \zeta}(a) + T_{\mathrm{III}}(a)\frac{\partial w_{\mathrm{III}}}{\partial \zeta}(a). \tag{9.38}$$

As depicted, at the outer points dampers are attached. Thus

$$T_k(b)\frac{\partial w_k}{\partial \zeta}(b) = -\alpha_k \frac{\partial w_k}{\partial t}(b), \qquad k = \mathrm{I}, \mathrm{II}, \text{ and III}. \tag{9.39}$$

(a) Assume that the constants α_I, α_{II} and α_{III} are non-negative. Show that then the coupled wave equation of Figure 9.3 generates a contraction semigroup on its energy space.

(b) Prove that the semigroup as defined in the previous item is exponentially stable if α_{I}, α_{II}, and α_{III} are positive.

9.4 Notes and references

Exponential stability of a system described by a partial differential equation is a well-studied topic. There are numerous papers on this subject, using different techniques. In the appendix of Cox and Zuazua [8], a simple proof is presented showing that the wave equation with damping at one boundary as in Example 9.2.1 is exponentially stable. It is shown in [59] that this idea can be applied to all (first order) port-Hamiltonian systems. The result as presented here follows [59].

Chapter 10

Inhomogeneous Abstract Differential Equations and Stabilization

In the previous five chapters we considered the homogeneous (abstract) differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{10.1}$$

However, for control theoretical questions it is essential to add an input to the differential equation, see e.g. Chapters 3 and 4. Section 10.1 is devoted to infinite-dimensional inhomogeneous differential equations and in Section 10.2 we add an output equation. The obtained formulas will be very similar to those found in Chapter 2. However, C_0-semigroups are in general not differentiable on an infinite-dimensional state space and thus the proofs are more involved.

As a control application we study in Section 10.4 the stabilization of an infinite-dimensional system by a finite-dimensional feedback. This result generalizes Theorem 4.3.3 to an infinite-dimensional state space. Since for stabilization of a finite-dimensional system we chose to work on $\mathbb{C}^n$, we assume throughout this chapter that our state space X is a Hilbert space over the complex numbers.

10.1 The abstract inhomogeneous Cauchy problem

If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$, then the classical solution of the abstract homogeneous Cauchy initial value problem

$$\dot{x}(t) = Ax(t), \; t \geq 0, \qquad x(0) = x_0 \in D(A)$$

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*,
Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_10,

is given by $x(t) = T(t)x_0$, see Lemma 5.3.2. In this section we consider the abstract inhomogeneous Cauchy problem

$$\dot{x}(t) = Ax(t) + f(t), \ t \geq 0, \qquad x(0) = x_0, \tag{10.2}$$

where for the moment we assume that $f \in C([0,\tau];X)$. (10.2) is also called an *abstract evolution equation* or *abstract differential equation*. First we define what we mean by a solution of (10.2), and we begin with the notion of a classical solution. $C^1([0,\tau];X)$ denotes the class of continuous functions on $[0,\tau]$ whose derivative is again continuous on $[0,\tau]$.

Definition 10.1.1. Consider equation (10.2) on the Hilbert space X and let $\tau > 0$. The function $x : [0,\tau] \to X$ is a *classical solution* of (10.2) *on* $[0,\tau]$ if $x \in C^1([0,\tau];X)$, $x(t) \in D(A)$ for all $t \in [0,\tau]$ and $x(t)$ satisfies (10.2) for all $t \in [0,\tau]$. The function x is a *classical solution on* $[0,\infty)$ if x is a classical solution on $[0,\tau]$ for every $\tau \geq 0$.

In the following we extend the finite-dimensional results of Section 2.2 to the infinite-dimensional situation.

Lemma 10.1.2. *Assume that $f \in C([0,\tau];X)$ and that $x : [0,\tau] \to X$ is a classical solution of* (10.2) *on* $[0,\tau]$. *Then $Ax(\cdot)$ is an element of $C([0,\tau];X)$, and we have*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \qquad t \in [0,\tau]. \tag{10.3}$$

Proof. From (10.2), we may conclude $Ax(t) = \dot{x}(t) - f(t)$ and $\dot{x} \in C([0,\tau];X)$ implies $Ax(\cdot) \in C([0,\tau];X)$.

Next we prove (10.3). Let t be an arbitrary, but fixed, element of $[0,\tau]$ and consider the function $s \mapsto T(t-s)x(s)$. We show that this function is differentiable on $[0,t)$. Let h be sufficiently small and consider

$$\frac{T(t-s-h)x(s+h) - T(t-s)x(s)}{h} = \frac{T(t-s-h)x(s+h) - T(t-s-h)x(s)}{h} + \frac{T(t-s-h)x(s) - T(t-s)x(s)}{h}.$$

If h converges to zero, then the second term on the right hand side converges to $-AT(t-s)x(s)$, since $x(s) \in D(A)$. Thus it remains to show that the first term on the right hand side converges. We have the equality

$$\begin{aligned}&\frac{T(t-s-h)x(s+h) - T(t-s-h)x(s)}{h} - T(t-s)\dot{x}(s) \\ &\quad = T(t-s-h)\left(\frac{x(s+h)-x(s)}{h} - \dot{x}(s)\right) + T(t-s-h)\dot{x}(s) - T(t-s)\dot{x}(s).\end{aligned}$$

The uniform boundedness of $(T(t))_{t\geq 0}$ on any compact interval, the differentiability of x and the strong continuity of $(T(t))_{t\geq 0}$ implies that the above expression converges to zero. Thus

$$\lim_{h\to 0} T(t-s-h)\frac{x(s+h)-x(s)}{h} = T(t-s)\dot{x}(s).$$

Thus the function $s \mapsto T(t-s)x(s)$ is differentiable on $[0,t)$, and

$$\frac{d}{ds}(T(t-s)x(s)) = -AT(t-s)x(s) + T(t-s)(Ax(s)+f(s)) = T(t-s)f(s),$$

where we used the equation $\dot{x}(s) = Ax(s)+f(s)$. Thus a classical solution to (10.2) necessarily has the form (10.3). □

Equation (10.3) is reminiscent of the variation of constants formula for ordinary differential equations, see (2.21). If x is a classical solution, then x is necessarily given by (10.3). Next we show that under suitable conditions (10.3) is already the (unique) classical solution of (10.2).

Theorem 10.1.3. *If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X, $f \in C^1([0,\tau];X)$ and $x_0 \in D(A)$, then the function $x : [0,\tau] \to X$ defined by* (10.3) *is continuously differentiable on* $[0,\tau]$ *and it is the unique classical solution of* (10.2).

Proof. Uniqueness: If x_1 and x_2 are two different classical solutions of (10.2), then their difference $\Delta(t) = x_1(t) - x_2(t)$ satisfies the differential equation

$$\frac{d\Delta}{dt} = A\Delta, \qquad \Delta(0) = 0$$

and so we need to show that its only solution is $\Delta(t) \equiv 0$. However, this follows directly from Lemma 5.3.2.

Existence: Clearly, all we need to show now is that $t \mapsto v(t) = \int_0^t T(t-s)f(s)ds$ is an element of $C^1([0,\tau];X)$, $v(t) \in D(A)$ for every $t \in [0,\tau]$ and v satisfies the differential equation (10.2) with $x_0 = 0$. Now

$$\begin{aligned} v(t) &= \int_0^t T(t-s)\left(f(0) + \int_0^s \dot{f}(\alpha)d\alpha\right) ds \\ &= \int_0^t T(t-s)f(0)ds + \int_0^t \int_\alpha^t T(t-s)\dot{f}(\alpha)dsd\alpha, \end{aligned} \tag{10.4}$$

where we have used Fubini's Theorem. From Theorem 5.2.2.5, it follows that the first term on the right hand side lies in the domain of A and that $\int_\alpha^t T(t-s)\dot{f}(\alpha)ds \in D(A)$ for all $\alpha \in [0,t]$. Furthermore, Theorem 5.2.2.5 implies

$$A\int_\alpha^t T(t-s)\dot{f}(\alpha)ds = T(t-\alpha)\dot{f}(\alpha) - \dot{f}(\alpha). \tag{10.5}$$

Since A is a closed operator, for every $g \in C([0,\tau],X)$ with values in $D(A)$ and $Ag \in C([0,\tau],X)$, we have $\int Ag(s)\,ds = A\int g(s)\,ds$. Combining this fact with (10.4) and (10.5), we obtain that $v(t) \in D(A)$, and

$$\begin{aligned}Av(t) &= (T(t)-I)f(0) + \int_0^t (T(t-\alpha)-I)\dot{f}(\alpha)d\alpha \\ &= T(t)f(0) + \int_0^t T(t-\alpha)\dot{f}(\alpha)d\alpha - f(t).\end{aligned}$$

Using the fact that the convolution product is commutative, i.e., $\int_0^t g(t-s)h(s)ds = \int_0^t g(s)h(t-s)ds$, we obtain

$$Av(t) = T(t)f(0) + \int_0^t T(\alpha)\dot{f}(t-\alpha)d\alpha - f(t).$$

Moreover, we have

$$\begin{aligned}T(t)f(0) + \int_0^t T(\alpha)\dot{f}(t-\alpha)d\alpha - f(t) &= \frac{d}{dt}\left(\int_0^t T(\alpha)f(t-\alpha)d\alpha\right) - f(t) \\ &= \dot{v}(t) - f(t),\end{aligned}$$

and thus we may conclude that $\dot{v}(t) = Av(t) + f(t)$. □

In the above proof, we have proved the following result.

Corollary 10.1.4. *If A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, and $f \in C^1([0,\tau];X)$, then the function $v : [0,\tau] \to X$ defined by $v(t) = \int_0^t T(t-s)f(s)ds$, $t \in [0,\tau]$ is continuously differentiable on $[0,\tau]$ and has values in the domain of A. Furthermore, it satisfies*

$$Av(t) = T(t)f(0) + \int_0^t T(t-\alpha)\dot{f}(\alpha)d\alpha - f(t). \tag{10.6}$$

The assumptions of Theorem 10.1.3 are too strong for control applications, where in general we do not wish to assume that $f \in C^1([0,\tau];X)$. Therefore we introduce the following weaker concept of a solution of (10.2).

Definition 10.1.5. If $f \in L^1([0,\tau];X)$ and $x_0 \in X$, then we call the function $x : [0,\tau] \to X$ defined by (10.3) a *mild solution* of (10.2) on $[0,\tau]$.

We note that (10.3) is a well-defined integral in the sense of Bochner or Pettis, (see Lemma A.1.6 and Example A.1.13). Of course, if $f \in L^p([0,\tau];X)$ for some $p \geq 1$, then necessarily $f \in L^1([0,\tau];X)$. In applications, we usually find $f \in L^2([0,\tau];X)$.

Lemma 10.1.6. *Assume that $f \in L^1([0,\tau];X)$ and $x_0 \in X$. The mild solution x defined by* (10.3) *is continuous on* $[0,\tau]$.

Proof. Since $T(\cdot)x_0$ is continuous, we can assume without loss of generality that $x_0 = 0$.

For $\delta > 0$, consider

$$\begin{aligned} x(t+\delta) - x(t) &= \int_0^t (T(t+\delta-s) - T(t-s))f(s)ds + \int_t^{t+\delta} T(t+\delta-s)f(s)ds \\ &= (T(\delta) - I)x(t) + \int_t^{t+\delta} T(t+\delta-s)f(s)ds. \end{aligned}$$

Then we estimate

$$\|x(t+\delta) - x(t)\| \le \|(T(\delta) - I)x(t)\| + \sup_{\alpha\in[0,\delta]} \|T(\alpha)\| \int_t^{t+\delta} \|f(s)\|ds$$

and the right-hand side converges to 0 as $\delta \downarrow 0$ by the strong continuity of the semigroup and Theorem 5.1.5.1. Now consider

$$x(t-\delta) - x(t) = \int_0^{t-\delta} (T(t-\delta-s) - T(t-s))f(s)ds - \int_{t-\delta}^t T(t-s)f(s)ds,$$

noting that $(T(t-\delta-s) - T(t-s))f(s)$ is integrable, since $f \in L^1([0,\tau];X)$ and using the properties of $(T(t))_{t\ge 0}$ from Theorem 5.1.5.1 and Example A.1.13. An estimation of the integral above yields

$$\|x(t-\delta) - x(t)\| \le \int_0^{t-\delta} \|(T(t-\delta-s) - T(t-s))f(s)\|ds + \int_{t-\delta}^t \|T(t-s)f(s)\|ds.$$

Now $(T(t-\delta-s) - T(t-s))f(s) \to 0$ as $\delta \downarrow 0$, and by Theorem 5.1.5 there exists a constant $M_t > 0$, depending only on t, such that $\|(T(t-\delta-s) - T(t-s))f(s)\| \le M_t\|f(s)\|$. So the first term converges to zero $\delta \downarrow 0$ by the Lebesgue Dominated Convergence Theorem, and the second term also tends to zero by similar arguments. □

In fact, the concept of a mild solution is the same as the concept of a *weak solution* used in the study of partial differential equations.

Definition 10.1.7. Let $f \in L^1([0,\tau];X)$. We call the function $x : [0,\tau] \to X$ a *weak solution* of (10.2) *on* $[0,\tau]$ if the following holds:

1. x is continuous on $[0,\tau]$;
2. For all $g \in C^1([0,\tau];X)$ with $g(t) \in D(A^*)$, $t \in [0,\tau]$ and $A^*g \in L^1([0,\tau];X)$ we have

$$\int_0^\tau \langle \dot g(t) + A^*g(t), x(t)\rangle dt + \int_0^\tau \langle g(t), f(t)\rangle dt = \langle g(t), x(\tau)\rangle - \langle g(0), x_0\rangle. \quad (10.7)$$

We call x a *weak solution* of (10.2) *on* $[0,\infty)$ if it is a weak solution on $[0,\tau]$ for every $\tau \geq 0$.

Theorem 10.1.8. *For every $x_0 \in X$ and every $f \in L^1([0,\tau];X)$ there exists a unique weak solution of* (10.2)*, which is given by the mild solution of* (10.2)*.*

Proof. Let $x_0 \in X$ and $f \in L^1([0,\tau];X)$ be chosen arbitrarily and let x be the corresponding mild solution given by (10.3). Lemma 10.1.6 shows that x is continuous on $[0,\tau]$. Let $g \in C^1([0,\tau];X)$ with $g(t) \in D(A^*)$, $t \in [0,\tau]$ and $A^*g \in L^1([0,\tau];X)$. It is sufficient to prove that the function $\langle g(\cdot), x(\cdot)\rangle$ is absolutely continuous and its derivative is given by

$$\frac{d\langle g(t), x(t)\rangle}{dt} = \langle A^* g(t), x(t)\rangle + \langle g(t), f(t)\rangle \tag{10.8}$$

for almost all t. It is easy to show that $\langle g(\cdot), x(\cdot)\rangle$ is absolutely continuous. For $h > 0$ we have that

$$\begin{aligned} &\langle g(t+h), x(t+h)\rangle - \langle g(t), x(t)\rangle \\ &= \langle g(t+h) - g(t), x(t+h)\rangle + \langle g(t), x(t+h) - x(t)\rangle. \end{aligned} \tag{10.9}$$

Since x is continuous and g is differentiable it is easy to see that

$$\lim_{h\to 0} \frac{\langle g(t+h) - g(t), x(t+h)\rangle}{h} = \langle \dot{g}(t), x(t)\rangle. \tag{10.10}$$

Furthermore, we have that

$$\begin{aligned} x(t+h) - x(t) &= x(t+h) - T(h)x(t) + T(h)x(t) - x(t) \\ &= \int_t^{t+h} T(t+h-s)f(s)ds + (T(h) - I)x(t). \end{aligned}$$

Let λ be an element of the resolvent set of A. By the previous equation, Proposition A.1.7, and Theorem 5.2.2 we have for almost all $t \in [0,\tau]$,

$$\begin{aligned} (\lambda I - A)^{-1} \lim_{h\downarrow 0} \frac{x(t+h) - x(t)}{h} &= (\lambda I - A)^{-1} \lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} T(t+h-s)f(s)ds \\ &\quad + \lim_{h\downarrow 0} \frac{(\lambda I - A)^{-1}(T(h) - I)x(t)}{h} \\ &= (\lambda I - A)^{-1} f(t) + \lim_{h\downarrow 0} \frac{(T(h) - I)(\lambda I - A)^{-1}x(t)}{h} \\ &= (\lambda I - A)^{-1} f(t) + A(\lambda I - A)^{-1}x(t). \end{aligned} \tag{10.11}$$

Combining (10.9), (10.10), and (10.11) we find that

$$\begin{aligned}
\lim_{h\downarrow 0}\frac{\langle g(t+h), x(t+h) - \langle g(t), x(t)\rangle}{h} &= \langle \dot{g}(t), x(t)\rangle + \lim_{h\downarrow 0}\langle g(t), x(t+h) - x(t)\rangle \\
&= \langle \dot{g}(t), x(t)\rangle + \lim_{h\downarrow 0}\langle (\lambda I - A)^* g(t), (\lambda I - A)^{-1}(x(t+h) - x(t))\rangle \\
&= \langle \dot{g}(t), x(t)\rangle + \langle (\lambda I - A)^* g(t), (\lambda I - A)^{-1} f(t) + A(\lambda I - A)^{-1} x(t)\rangle \\
&= \langle \dot{g}(t), x(t)\rangle + \langle g(t), f(t)\rangle + \langle A^* g(t), x(t)\rangle.
\end{aligned}$$

This proves (10.8) for the derivative from the right. The proof for the derivative from the left goes similarly, see also the proof of Lemma 10.1.2.

To prove the uniqueness, we use the fact that A^* is the infinitesimal generator of the C_0-semigroup $(T^*(t))_{t\geq 0}$, see e.g. [10, 15, 24, 61].

If there are two weak solutions, then the difference $x_d(t)$ satisfies

$$\int_0^\tau \langle \dot{g}(t) + A^* g(t), x_d(t)\rangle dt = \langle g(\tau), x_d(\tau)\rangle \tag{10.12}$$

for all $g \in C^1([0,\tau], X)$ with $g(t) \in D(A^*)$, $t \in [0,\tau]$ and $A^* g \in L^1([0,\tau]; X)$. Take $h \in C^1([0,\tau], X)$, and define z as the unique classical solution of

$$\dot{z}(t) = A^* z(t) + h(t), \qquad z(0) = 0.$$

By Lemma 10.1.2 this solution exists. We define $g(t) = z(\tau - t), t \in [0,\tau]$. Using Lemma 10.1.2, it is easy to see that $g \in C^1([0,\tau], X)$ and $g(t) \in D(A^*)$ for all $t \in [0,\tau]$. Furthermore,

$$\dot{g}(t) = -\dot{z}(\tau - t) = -A^* z(\tau - t) - h(\tau - t), \qquad g(\tau) = 0.$$

Substituting this into equation (10.12), we obtain

$$-\int_0^\tau \langle h(\tau - t), x_d(t)\rangle dt = 0.$$

As this hold for all continuous functions h, and since x_d is continuous, we conclude that x_d is the zero function. □

In the following chapters, whenever we refer to the solution of the abstract evolution equation (10.2), we mean the mild solution (10.3).

Example 10.1.9. In this example, we shall again consider the heat equation of Example 5.1.1. The model of the heated bar was given by

$$\begin{aligned}
&\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + u(\zeta, t), \quad x(\zeta, 0) = x_0(\zeta), \\
&\frac{\partial x}{\partial \zeta}(0, t) = 0 = \frac{\partial x}{\partial \zeta}(1, t).
\end{aligned}$$

We showed in Section 5.3 that if $u = 0$, then this partial differential equation can be formulated as an abstract differential equation on $X = L^2(0,1)$ of the form

$$\dot{x}(t) = Ax(t),\ t \geq 0, \qquad x(0) = x_0,$$

where

$$Ah = \frac{d^2h}{d\zeta^2} \quad \text{with} \tag{10.13}$$

$$D(A) = \Big\{ h \in L^2(0,1) \mid h, \frac{dh}{d\zeta} \text{ are absolutely continuous,}$$
$$\frac{d^2h}{d\zeta^2} \in L^2(0,1) \text{ and } \frac{dh}{d\zeta}(0) = 0 = \frac{dh}{d\zeta}(1) \Big\}. \tag{10.14}$$

We can include the control term in this formulation as follows:

$$\dot{x}(t) = Ax(t) + u(t),\ t \geq 0, \qquad x(0) = x_0,$$

provided that $u \in L^1([0,\tau]; L^2(0,1))$. The solution is given by (10.3), which can be written as, see Example 5.1.4,

$$\begin{aligned} x(\zeta,t) &= \sum_{n=0}^{\infty} e^{\lambda_n t} \langle x_0, \phi_n\rangle \phi_n(\zeta) + \int_0^t \sum_{n=0}^{\infty} e^{\lambda_n (t-s)} \langle u(\cdot,s), \phi_n(\cdot)\rangle \phi_n(\zeta) ds \\ &= \int_0^1 z_0(\alpha) d\alpha + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \int_0^1 z_0(\alpha)\cos(n\pi\alpha) d\alpha \cos(n\pi\zeta) \\ &\quad + \int_0^t \int_0^1 u(\alpha,s) d\alpha + \int_0^t \sum_{n=1}^{\infty} e^{-n^2\pi^2(t-s)} 2 \int_0^1 u(y,s)\cos(n\pi\alpha) d\alpha \cos(n\pi\zeta) ds, \end{aligned}$$

since $\lambda_n = -n^2\pi^2, n \geq 0$, $\phi_n(\zeta) = \sqrt{2}\cos(n\pi\zeta), n \geq 1$ and $\phi_0(\zeta) = 1$.

10.2 Outputs

Usually, as in Chapter 2, the function f in the inhomogeneous abstract differential equation (10.2) is of the form $Bu(\cdot)$, where B is a bounded linear operator from the *input space* U to the state space X. We assume that U is also a Hilbert space. Thus the equation (10.2) reads

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0. \tag{10.15}$$

Again we assume that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$. The state is x and the input is given by u. We denote the system (10.15) by $\Sigma(A,B)$.

In many systems the state is measured via an output, and so we add an output to the system $\Sigma(A, B)$. We start by assuming that the output equation can be represented via a bounded operator. For the more general case, we refer to section 11.2.

The system $\Sigma(A, B)$ with an *output* equation is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{10.16}$$

$$y(t) = Cx(t) + Du(t), \tag{10.17}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on the state space X, B is a linear bounded operator from the input space U to X, C is a linear bounded operator from X to the *output space* Y, and D is a linear operator from U to Y. All spaces U, X and Y are supposed to be Hilbert spaces. We denote the system (10.16)–(10.17) by $\Sigma(A, B, C, D)$.

In Section 10.1 we showed that the mild solution of (10.16) is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\, ds.$$

This function is well-defined for every $x_0 \in X$ and every $u \in L^1([0, \tau]; U)$, $\tau > 0$. As the operators C and D are bounded, there is no difficulty in "solving" equation (10.17). We summarize the answer in the following theorem.

Theorem 10.2.1. *Consider the abstract equation* (10.16)–(10.17)*, with A the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$, and B, C, and D bounded linear operators. The mild solution of* (10.16)–(10.17) *is given by the variation of constant formula* (10.3)

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\, ds,$$

$$y(t) = CT(t)x_0 + C\int_0^t T(t-s)Bu(s)\, ds + Du(t)$$

for every $x_0 \in X$ and every $u \in L^1([0, \tau]; U)$.

As an example we return to Example 10.1.9 to which we add a measurement.

Example 10.2.2. Consider the heated bar of Example 10.1.9 on which we measure the average temperature in the first half of the bar.

$$\begin{aligned}
&\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + u(\zeta, t), \quad x(\zeta, 0) = x_0(\zeta),\\
&\frac{\partial x}{\partial \zeta}(0, t) = 0 = \frac{\partial x}{\partial \zeta}(1, t),\\
&y(t) = \textstyle\int_0^{\frac{1}{2}} x(\zeta, t)\, d\zeta.
\end{aligned}$$

From Example 10.1.9 we have that the infinitesimal generator is given by (10.13) and (10.14). Furthermore, it is easy to see that the input operator equals the identity, i.e., $U = X$ and $B = I$. The output space equals $\mathbb{C}$ and the output operator C is given by

$$Cx = \int_0^{\frac{1}{2}} x(\zeta)\, d\zeta. \tag{10.18}$$

This is clearly a bounded linear operator, and so this partial differential equation can be written in the form (10.16)–(10.17).

10.3 Bounded perturbations of C_0-semigroups

In applications to control problems, the inhomogeneous term f in (10.2) is often determined by a control input of feedback type, namely,

$$f(t) = Dx(t),$$

where $D \in \mathcal{L}(X)$. This leads to the new Cauchy problem

$$\dot{x}(t) = (A + D)x(t), \ t \geq 0, \qquad x(0) = x_0, \tag{10.19}$$

or in its integrated form

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Dx(s)ds. \tag{10.20}$$

We expect that the perturbed system operator, $A+D$, is the infinitesimal generator of another C_0-semigroup $(T_D(t))_{t\geq 0}$, such that the solution of (10.19) is given by $x(t) = T_D(t)x_0$. This result is proved in Exercise 10.1 for the case that A generates a contraction semigroup. For the general case, we refer to [10, Section 3.2] or [15, page 158].

Theorem 10.3.1. *Suppose that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X and that $D \in \mathcal{L}(X)$. Then $A+D$ is the infinitesimal generator of a C_0-semigroup $(T_D(t))_{t\geq 0}$ on X. Moreover, if $\|T(t)\| \leq Me^{\omega t}$, then*

$$\|T_D(t)\| \leq Me^{(\omega + M\|D\|)t} \tag{10.21}$$

and for every $x_0 \in X$ the following equations are satisfied:

$$T_D(t)x_0 = T(t)x_0 + \int_0^t T(t-s)DT_D(s)x_0 ds \tag{10.22}$$

and

$$T_D(t)x_0 = T(t)x_0 + \int_0^t T_D(t-s)DT(s)x_0 ds. \tag{10.23}$$

10.4 Exponential stabilizability

In Chapter 4 we studied the stabilizability of the finite-dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0.$$

In Sections 10.1 and 10.2 we saw that this equation is well-defined provided A is the infinitesimal generator of a C_0-semigroup and B is a bounded operator. Thus the concept of stabilizability generalizes naturally to an infinite-dimensional setting.

Definition 10.4.1. Suppose that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X and that $B \in \mathcal{L}(U, X)$, where U is a Hilbert space. If there exists an $F \in \mathcal{L}(X, U)$ such that $A + BF$ generates an exponentially stable C_0 -semigroup, $(T_{BF}(t))_{t\geq 0}$, then we say that $\Sigma(A, B)$ is *exponentially stabilizable*.

An operator $F \in \mathcal{L}(X, U)$ will be called a *feedback operator*.

Remark 10.4.2. This definition differs from our definition of exponential stabilizability for finite-dimensional systems, see Definition 4.1.3. By Theorem 4.3.1, for finite-dimensional systems Definitions 4.1.3 and 10.4.1 are equivalent. For infinite-dimensional systems this no longer holds, see also the Notes and References.

As for finite-dimensional state spaces, we aim to characterize operators A and Bs such that $\Sigma(A, B)$ is stabilizable. Unfortunately, such a characterization is not known. However, if the input space is finite-dimensional, or a little bit more general if the range of B is finite-dimensional, then a complete characterization of all systems $\Sigma(A, B)$ which are exponentially stabilizable can be given. In this section we present this result. Hence we assume that $B \in \mathcal{L}(\mathbb{C}^m, X)$, and that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X.

For the proof of the main theorem we need the following lemma. Note that the order ν_0 of the isolated eigenvalue λ_0 is defined as follows. For every $x \in X$, $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0} (\lambda I - A)^{-1} x$ exists, but there exists an $x_0 \in X$ such that $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{\nu_0 - 1} (\lambda I - A)^{-1} x_0$ does not exist, see also page 103. For the eigenvalue λ_0 of the operator Q with order $\nu_0 < \infty$ the *multiplicity* is defined as $\dim \ker \left((\lambda_0 I - Q)^{\nu_0} \right)$.

Lemma 10.4.3. *Consider the system $\Sigma(A, B)$ with $B \in \mathcal{L}(\mathbb{C}^m, X)$ and let $F \in \mathcal{L}(X, \mathbb{C}^m)$. For $s \in \rho(A + BF)$ we have the following properties:*

1. $-1 \in \sigma(BF(sI - A - BF)^{-1})$ *if and only if* $-1 \in \sigma(F(sI - A - BF)^{-1}B)$ *if and only if* $\det(I + F(sI - A - BF)^{-1}B) = 0$.

    If $-1 \in \sigma(BF(sI - A - BF)^{-1})$, then -1 lies in the point spectrum. The same assertion holds for $F(sI - A - BF)^{-1}B$.

2. *The order and the multiplicity of the eigenvalue -1 of $BF(sI - A - BF)^{-1}$ and $F(sI - A - BF)^{-1}B$ are finite and equal.*

3. *$s \in \rho(A)$ if and only if $-1 \in \rho(F(sI - A - BF)^{-1}B)$. If $s \in \rho(A)$, then $(sI - A)^{-1}$ satisfies*

$$(sI - A)^{-1} = (sI - A - BF)^{-1} - (sI - A - BF)^{-1}B \cdot \left(I + F(sI - A - BF)^{-1}B\right)^{-1} F(sI - A - BF)^{-1}. \quad (10.24)$$

4. *If the holomorphic function $s \mapsto \det(I+F(sI-A-BF)^{-1}B)$, $s \in \rho(A+BF)$, is zero for $s = s_0$, but not identically zero in a neighbourhood of s_0, then $s_0 \in \sigma(A)$ and it is an eigenvalue of A with finite order and finite multiplicity.*

Proof. 1. The proof is based on the following equalities;

$$\left(BF(sI - A - BF)^{-1}\right) B = B\left(F(sI - A - BF)^{-1}B\right) \qquad \text{and} \qquad (10.25)$$

$$F(sI - A - BF)^{-1}\left(BF(sI - A - BF)^{-1}\right) = \left(F(sI - A - BF)^{-1}B\right) F(sI - A - BF)^{-1}. \qquad (10.26)$$

Since $F(sI-A-BF)^{-1}B$ is an $m\times m$-matrix all its spectral points are eigenvalues. Furthermore, since the range of the bounded linear operator $BF(sI - A - BF)^{-1}$ is contained in $\mathbb{C}^m$, it is a compact operator, and so every non-zero point in the spectrum is an isolated eigenvalue. This proves the second part of the first assertion. Let -1 be an eigenvalue of $F(sI - A - BF)^{-1}B$ and let $v \in \mathbb{C}^m$ be the corresponding eigenvector, i.e,

$$F(sI - A - BF)^{-1}Bv = -v.$$

Thus in particular $Bv \neq 0$, and by multiplying this equation with B from the left and using (10.25) it follows that Bv is an eigenvector of $BF(sI - A - BF)^{-1}$ with eigenvalue -1. The other implication can be proved in a similar manner.

2. From the previous part it follows that the dimension of the kernel of $I + BF(sI - A - BF)^{-1}$ equals the dimension of the kernel of $I + F(sI - A - BF)^{-1}B$. Similarly the dimensions of the kernel of $\left(I + BF(sI - A - BF)^{-1}\right)^{\nu}$ and $\left(I + F(sI - A - BF)^{-1}B\right)^{\nu}$ are the same. Furthermore, since $F(sI - A - BF)^{-1}B$ is an $m \times m$ and $BF(sI - A - BF)^{-1}B$ is a compact operator the eigenvalue -1 is for both operators of finite order. Combining these two facts, we conclude that they have equal (finite) order and multiplicity.

3. We have the following simple equality:

$$sI - A = \left(I + BF(sI - A - BF)^{-1}\right)(sI - A - BF). \qquad (10.27)$$

Since $sI - A - BF$ is boundedly invertible, the operator $sI - A$ is injective if and only if $I + BF(sI - A - BF)^{-1}$ is injective. Using the same argument, we obtain that $sI - A$ is surjective if and only if $I + BF(sI - A - BF)^{-1}$ is surjective. Using the fact that $BF(sI-A-BF)^{-1}$ is a compact operator, $I+BF(sI-A-BF)^{-1}$ is injective if and only if it is surjective if and only if $-1 \in \rho(BF(sI - A - BF)^{-1})$. In particular, this implies that $s \in \rho(A)$ if and only if $-1 \in \rho(F(sI - A - BF)^{-1}B)$. Hence we have proved the first part. It is easy to see that the following equalities hold for $s \in \rho(A + BF)$:

$$I = (sI - A)(sI - A - BF)^{-1} - BF(sI - A - BF)^{-1}, \tag{10.28}$$

$$B = (sI - A)(sI - A - BF)^{-1}B - BF(sI - A - BF)^{-1}B, \tag{10.29}$$

and hence

$$B\left(I + F(sI - A - BF)^{-1}B\right) = (sI - A)(sI - A - BF)^{-1}B. \tag{10.30}$$

Thus for $s \in \rho(A) \cap \rho(A + BF)$ we have

$$(sI - A)^{-1}B = (sI - A - BF)^{-1}B\left(I + F(sI - A - BF)^{-1}B\right)^{-1}. \tag{10.31}$$

Multiplying (10.28) from the left by $(sI - A)^{-1}$ and using (10.31) we find that

$$\begin{aligned}(sI - A)^{-1} &= (sI - A - BF)^{-1} - (sI - A)^{-1}BF(sI - A - BF)^{-1} \\ &= (sI - A - BF)^{-1} - (sI - A - BF)^{-1}B \\ &\quad \cdot \left(I + F(sI - A - BF)^{-1}B\right)^{-1} F(sI - A - BF)^{-1}.\end{aligned}$$

Thus we have shown (10.24).

4. Since $\det(I + F(sI - A - BF)^{-1}B)$ is not identically zero in a neighbourhood of s_0, part 1 implies that s_0 is an isolated eigenvalue of A. It remains to show that the order and multiplicity of $s_0 \in \sigma(A)$ are finite. Let ν_0 be the order of s_0 as a zero of $\det(I + F(sI - A - BF)^{-1}B)$. Using (10.24) we find that

$$\begin{aligned}\lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A)^{-1}x &= \lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A - BF)^{-1}x \\ &\quad - \lim_{s \to s_0}(s - s_0)^{\nu_0}(sI - A - BF)^{-1}B \\ &\quad \cdot \left(I + F(sI - A - BF)^{-1}B\right)^{-1} F(sI - A - BF)^{-1}x.\end{aligned}$$

By the definition of ν_0 and the fact that $(sI - A - BF)^{-1}$ is holomorphic on a neighbourhood of s_0, the limit on the right hand side exists. Hence the order of s_0 as an eigenvalue of A cannot be larger than ν_0. That the multiplicity is finite follows from (10.27), see also part 1. □

Lemma 10.4.3 implies that if $s \in \rho(A + BF) \cap \sigma(A)$, then s is an eigenvalue of A. In fact, we show that if $\Sigma(A, B)$ is stabilizable then A can have at most finitely many eigenvalues in $\overline{\mathbb{C}_0^+}$. Such a separation of the spectrum is an important property of the generator. We introduce the following notation:

$$\sigma^+ := \sigma(A) \cap \overline{\mathbb{C}_0^+};\ \mathbb{C}_0^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 0\}, \tag{10.32}$$

$$\sigma^- := \sigma(A) \cap \mathbb{C}_0^-;\ \mathbb{C}_0^- = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) < 0\}. \tag{10.33}$$

Definition 10.4.4. A satisfies the *spectrum decomposition assumption at zero* if σ^+ is bounded and separated from σ^- in such a way that a rectifiable, simple, closed curve, Γ, can be drawn so as to enclose an open set containing σ^+ in its interior and σ^- in its exterior.

From Theorem 8.2.4 follows that if the spectral decomposition assumption at zero holds, then we have a corresponding decomposition of the state space X and of the operator, A. More precisely, the spectral projection P_Γ defined by

$$P_\Gamma x = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} x d\lambda, \tag{10.34}$$

where Γ is traversed once in the positive direction (counterclockwise), induces the following decomposition:

$$X = X^+ \oplus X^-, \qquad \text{where } X^+ := P_\Gamma X \text{ and } X^- := (I - P_\Gamma)X. \tag{10.35}$$

In view of this decomposition, it is convenient to use the notation

$$A = \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix}, \quad T(t) = \begin{bmatrix} T^+(t) & 0 \\ 0 & T^-(t) \end{bmatrix}, \quad B = \begin{bmatrix} B^+ \\ B^- \end{bmatrix}, \tag{10.36}$$

where $B^+ = P_\Gamma B \in \mathcal{L}(U, X^+)$, and $B^- = (I - P_\Gamma)B \in \mathcal{L}(U, X^-)$. In fact, we have decomposed our system $\Sigma(A, B)$ as the vector sum of the two subsystems: $\Sigma(A^+, B^+)$ on X^+ and $\Sigma(A^-, B^-)$ on X^-.

The following theorem shows that if the system $\Sigma(A, B)$ is stabilizable, then X^+ is finite-dimensional, and $(T^-(t))_{t\geq 0}$ is exponentially stable.

Theorem 10.4.5. *Consider the system $\Sigma(A, B)$ on the state space X and input space $\mathbb{C}^m$. The following assertions are equivalent:*

1. $\Sigma(A, B)$ *is exponentially stabilizable;*
2. $\Sigma(A, B)$ *satisfies the spectrum decomposition assumption at zero, X^+ is finite-dimensional, $(T^-(t))_{t\geq 0}$ is exponentially stable, and the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable, where we have used the notation introduced in equations* (10.35) *and* (10.36).

If $\Sigma(A, B)$ is exponentially stabilizable, then a stabilizing feedback operator is given by $F = F^+ P_\Gamma$, where F^+ is a stabilizing feedback operator for $\Sigma(A^+, B^+)$.

Proof. $2 \Rightarrow 1$. Since the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable, there exists a feedback operator $F^+ \in \mathcal{L}(X^+, \mathbb{C}^m)$ such that the spectrum of $A^+ + B^+F^+$ lies in $\mathbb{C}_0^-$. Choose the feedback operator $F = [F^+ \; 0] \in \mathcal{L}(X, \mathbb{C}^m)$ for the system $\Sigma(A, B)$. The perturbed operator $A + BF = \begin{bmatrix} A^+ + B^+F^+ & 0 \\ B^-F^+ & A^- \end{bmatrix}$ generates a C_0-semigroup by Theorem 10.3.1. Furthermore, for $s \in \overline{\mathbb{C}_0^+}$ we have that, see Exercise 10.2,

$$(sI - A - BF)^{-1} = \begin{bmatrix} (sI - A^+ - B^+F^+)^{-1} & 0 \\ (sI - A^-)^{-1} B^- F^+ (sI - A^+ - B^+F^+)^{-1} & (sI - A^-)^{-1} \end{bmatrix}.$$

Since the semigroups generated by $A^+ + B^+F^+$ and A^- are exponentially stable, Theorem 8.1.4 implies that the corresponding resolvent operators are uniformly bounded in the right half-plane. Thus $(sI - A - BF)^{-1}$ is uniformly bounded in $\mathbb{C}_0^+$, and thus by Theorem 8.1.4 its corresponding semigroup is exponentially stable.

$1 \Rightarrow 2$. By assumption there exists an operator $F \in \mathcal{L}(X, \mathbb{C}^m)$ such that $A + BF$ generates an exponentially stable C_0-semigroup. By Definition 8.1.1, there exist constants $M > 0$ and $\gamma < 0$, such that

$$\|T_{BF}(t)\| \leq M e^{\gamma t}. \tag{10.37}$$

From Lemma 10.4.3, for every $s \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \gamma\}$ we have $s \in \sigma(A)$ if and only if $\det(I + F(sI - A - BF)^{-1}B) = 0$. Now the determinant $\det(I + F(\cdot I - A - BF)^{-1}B)$ is holomorphic on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \gamma\}$ and therefore there cannot be an accumulation point of zeros in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \gamma + \varepsilon\}$, $\varepsilon > 0$, unless the determinant is identically zero.

From (10.37), it follows that for all $\varepsilon > 0$,

$$\int_0^\infty e^{2(-\gamma-\varepsilon)t} \|F T_{BF}(t) B\|^2 dt < \infty$$

and by the Paley-Wiener Theorem A.2.9, we deduce that

$$F((\cdot + \gamma + \varepsilon)I - A - BF)^{-1}B \in \mathbf{H}^2(\mathbb{C}^{m\times m}). \tag{10.38}$$

By Lemma A.2.6 this implies that

$$\lim_{\rho\to\infty} \sup_{s\in\mathbb{C}^+_{\gamma+\varepsilon}, |s|\geq\rho} \|F(sI - A - BF)^{-1}B\| = 0.$$

Consequently, $\det(I + F(\cdot I - A - BF)^{-1}B)$ cannot be identically zero in $\overline{\mathbb{C}_0^+}$, and the function has no finite accumulation point there. Moreover, we can always find a sufficiently large ρ such that

$$\|F(\cdot I - A - BF)^{-1}B\| \leq \frac{1}{2} \text{ in } \overline{\mathbb{C}_0^+} \backslash \mathbb{D}(\rho), \tag{10.39}$$

where $\mathbb{D}(\rho) = \{s \in \overline{\mathbb{C}_0^+} \mid |s| \leq \rho\}$. Thus $I + F(sI - A - BF)^{-1}B$ is invertible for all $s \in \overline{\mathbb{C}_0^+} \setminus \mathbb{D}(\rho)$. Inside the compact set $\mathbb{D}(\rho)$ a holomorphic function has at most finitely many zeros, and applying Lemma 10.4.3 we see that σ^+ comprises at most finitely many points with finite multiplicity. Hence the spectrum decomposition assumption holds at zero. From Theorem 8.2.4.4 and e follows that $X^+ = \operatorname{ran} P^+$ is finite-dimensional and $\sigma(A^+) = \sigma^+ \subset \overline{\mathbb{C}_0^+}$. Thus it remains to show that $(T^-(t))_{t\geq 0}$ is exponentially stable and that $\Sigma(A^+, B^+)$ is controllable.

By Lemma 10.4.3 we have that

$$(sI - A)^{-1} = (sI - A - BF)^{-1} - (sI - A - BF)^{-1}B \cdot \left(I + F(sI - A - BF)^{-1}B\right)^{-1} F(sI - A - BF)^{-1}. \tag{10.40}$$

Since $A + BF$ generates an exponentially stable semigroup, by Theorem 8.1.4 we obtain that $(\cdot I - A - BF)^{-1}$ is uniformly bounded in $\mathbb{C}_0^+$. Combining this with (10.39), equation (10.40) implies that

$$\sup_{s \in \overline{\mathbb{C}_0^+} \setminus \mathbb{D}(\rho)} \|(sI - A)^{-1}\| < \infty.$$

From Lemma 8.2.6 we conclude that $(T^-(t))_{t \geq 0}$ is exponentially stable.

Finally, we prove that the system $\Sigma(A^+, B^+)$ is controllable. By Theorem 10.3.1 we obtain that

$$P_\Gamma T_{BF}(t)x_0 = P_\Gamma T(t)x_0 + \int_0^t P_\Gamma T(t-s)BFT_{BF}(s)x_0 ds.$$

For $x_0 \in X^+$ this equation reads

$$P_\Gamma T_{BF}(t)x_0 = T^+(t)x_0 + \int_0^t T^+(t-s)B^+FT_{BF}(s)x_0 ds.$$

Since $(T_{BF}(t))_{t \geq 0}$ is exponentially stable, for any $x_0 \in X^+$ there exists an input $u \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{C}^m)$, i.e., $u(t) = FT_{BF}(t)x_0$, such that the solution of the finite-dimensional system

$$\dot{x}(t) = A^+x(t) + B^+u(t), \qquad x(0) = x_0$$

is exponentially decaying. By Definition 4.1.3, the system $\Sigma(A^+, B^+)$ is stabilizable. Since all eigenvalues of the matrix A^+ lie in $\overline{\mathbb{C}_0^+}$, using Theorem 4.3.3 we may conclude that $\Sigma(A^+, B^+)$ is controllable. □

We apply the above result to the system of Example 10.1.9.

Example 10.4.6. Here we consider again the heat equation of Example 10.1.9, but we choose a different input operator.

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + \,_{[\frac{1}{2},1]}(\zeta)u(t), \qquad x(\zeta, 0) = x_0(\zeta), \tag{10.41}$$

$$\frac{\partial x}{\partial \zeta}(0, t) = 0 = \frac{\partial x}{\partial \zeta}(1, t). \tag{10.42}$$

As in Example 10.1.9 we can write this partial differential equation as an abstract differential equation on $X = L^2(0,1)$:

$$\dot{x}(t) = Ax(t) + Bu(t), \; t \geq 0, \qquad x(0) = x_0,$$

where A is given by (10.13) and (10.14) and $B \in \mathcal{L}(\mathbb{C}, X)$ is given by

$$(Bu)(\zeta) = \ _{[\frac{1}{2},1]}(\zeta)u.$$

Since $\sigma(A) \cap \overline{\mathbb{C}_0^+} = \{0\}$, the system is not exponentially stable. Using the fact that X possesses an orthonormal basis of eigenvectors of A, see Example 5.1.4, it is easy to see that

$$P_\Gamma x = \langle x, \phi_0 \rangle \phi_0$$

and thus $X^+ = \text{span}\{\phi_0\}$, $X^- = \overline{\text{span}_{n\geq 1}}\{\phi_n\}$. Furthermore, the semigroup $(T^-(t))_{t\geq 0}$ is given by

$$T^-(t)x = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \langle x, \phi_n \rangle \phi_n$$

and therefore the semigroup $(T^-(t))_{t\geq 0}$ is exponentially stable. If we can show that $\Sigma(A^+, B^+)$ is controllable, then Theorem 10.4.5 implies that the system (10.41)–(10.42) is exponentially stabilizable. We have that $A^+ = 0$ and $B^+ = P_\Gamma B = \langle \ _{[\frac{1}{2},1]}, 1\rangle 1 = \frac{1}{2}$. Using Theorem 3.1.6 it is easy to see that the one-dimensional system $\Sigma(0, \frac{1}{2})$ is controllable, and therefore the controlled heat equation is exponentially stabilizable.

10.5 Exercises

10.1. In this exercise we prove Theorem 10.3.1 in the case that A generates a contraction semigroup. Hence throughout this exercise we assume that A is the infinitesimal generator of the contraction semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X.

(a) Show that the operator $A + D - 2\|D\|I$ with domain $D(A)$ generates a contraction semigroup on X.
Use Exercise 5.3 to conclude that $A + D$ with domain $D(A)$ generates a C_0-semigroup on X.

(b) Denote by $(T_D(t))_{t\geq 0}$ the C_0-semigroup generated by $A + D$.
Let $x_0 \in D(A)$. Use Theorem 10.1.3 and show that $x(t) := T_D(t)x_0 - T(t)x_0 - \int_0^t T(t-s)DT_D(s)x_0\, ds$ is a classical solution of

$$\dot{x}(t) = Ax(t), \qquad x(0) = 0.$$

Use Lemma 5.3.2 to conclude that $x(t) = 0$ for $t \geq 0$.

(c) Let $x_0 \in D(A)$. Show that $x(t) = T(t)x_0 + \int_0^t T_D(t-s)DT(s)x_0\, ds$ is a classical solution of

$$\dot{x}(t) = (A + D)x(t), \qquad x(0) = x_0.$$

(d) Prove that (10.22) and (10.23) hold.

10.2. Let A_1 and A_2 be two closed, densely defined operators on X_1 and X_2, respectively. Let Q be a bounded operator from X_1 to X_2. Show that if A_1 and A_2 are (boundedly) invertible, then the following operator

$$A_{\text{ext}} = \begin{bmatrix} A_1 & 0 \\ Q & A_2 \end{bmatrix}, \qquad D(A_{\text{ext}}) = D(A_1) \oplus D(A_2),$$

is invertible as well, and the inverse is given by

$$A_{\text{ext}}^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_2^{-1} Q A_1^{-1} & A_2^{-1} \end{bmatrix}.$$

10.3. Consider the partial differential equation

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta, t) + 10x(\zeta, t) + {}_{[0,\frac{1}{4}]}(\zeta)u(t), \qquad x(\zeta, 0) = x_0(\zeta), \tag{10.43}$$

$$\frac{\partial x}{\partial \zeta}(0, t) = 0 = \frac{\partial x}{\partial \zeta}(1, t) \tag{10.44}$$

$$y(t) = \int_0^1 x(\zeta, t)\, d\zeta - u(t). \tag{10.45}$$

(a) Formulate the partial differential equation (10.43)–(10.45) as a system of the form (10.16)–(10.17) on the state $X = L^2(0, 1)$.

(b) Is the system exponentially stabilizable?

10.6 Notes and references

The results as formulated in Section 10.1 and 10.3 are well-documented in the literature and can be found in any standard text on C_0-semigroups, such as [24], [61], and [15]. However, since these results are of eminent importance for applications, textbooks aimed on applications of semigroup theory contain these results as well, see [10] or [44].

Since the output equation is important for control theory only, the results of Section 10.2 are not treated in books on C_0-semigroups. For more results on this, and on the state space system $\Sigma(A, B, C, D)$ we refer to [10].

In our definition of exponential stabilizability, Definition 10.4.1, we have chosen the control as $u(t) = Fx(t)$. For the finite-dimensional case we could equivalently allow for locally L^1-functions. This equivalence no longer holds for infinite-dimensional systems, see Example 2.13 of [48]. However, if the (stabilizing) input u is chosen as an $L^2([0, \infty); U)$-function, then there exists a bounded F such that $A + BF$ generates an exponentially stable semigroup. The proof uses optimal control theory, see e.g. Chapter 6 of [10].

The implication 2 to 1 in Theorem 10.4.5 has been known since the mid 1970s of the last century. However, the reserve implication stayed open for at least ten years. Then it was proved independently of each other by Desch and Schappacher [11], Jacobson and Nett [27], and Nefedov and Sholokhovich [42].

Chapter 11

Boundary Control Systems

In this chapter we are in particular interested in systems with a control at the boundary of their spatial domain. We show that these systems have well-defined solutions provided the input is sufficiently smooth. The simplest boundary control system is most likely the controlled transport equation, which is given by

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ x(1,t) &= u(t), && t \geq 0, \end{aligned} \tag{11.1}$$

where u denotes the control function. In Exercise 6.3 we have solved this partial differential equation for the specific choice $u = 0$. In this chapter we show that the solution of (11.1) is given by

$$x(\zeta,t) = \begin{cases} x_0(\zeta+t), & \zeta+t \leq 1, \\ u(\zeta+t-1), & \zeta+t > 1. \end{cases}$$

11.1 Boundary control systems

Boundary control problems like (11.1) occur frequently in applications, but unfortunately they do not fit into our standard formulation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \tag{11.2}$$

However, for sufficiently smooth inputs it is possible to reformulate these problems in such a way that they lead to an associated system in the standard form (11.2).

First we explain the idea behind this reformulation for the system (11.1). Assume that x is a classical solution of the p.d.e. (11.1) and that u is continuously differentiable. Defining

$$v(\zeta,t) = x(\zeta,t) - u(t),$$

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Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_11,

we obtain the following partial differential equation for v

$$\frac{\partial v}{\partial t}(\zeta,t) = \frac{\partial v}{\partial \zeta}(\zeta,t) - \dot{u}(t), \qquad \zeta \in [0,1],\ t \geq 0$$
$$v(1,t) = 0, \qquad t \geq 0.$$

This partial differential equation for v can be written in the standard form as

$$\dot{v}(t) = Av(t) + B\tilde{u}(t)$$

for $\tilde{u} = \dot{u}$. Hence via a simple trick, we can reformulate this p.d.e. with boundary control into a p.d.e. with internal control. The price we have to pay is that u has to be smooth.

The trick applied to (11.1) can be extended to abstract control systems of the form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t), \end{aligned} \tag{11.3}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$ is linear, the control function u takes values in the Hilbert space U, and the *boundary operator* $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$ is linear and satisfies $D(\mathfrak{A}) \subset D(\mathfrak{B})$.

In order to reformulate equation (11.3) into an abstract form (11.2), we need to impose extra conditions on the system.

Definition 11.1.1. The control system (11.3) is a *boundary control system* if the following hold:

1. The operator $A : D(A) \to X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and
$$Ax = \mathfrak{A}x \qquad \text{for } x \in D(A) \tag{11.4}$$
is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X;
2. There exists an operator $B \in \mathcal{L}(U,X)$ such that for all $u \in U$ we have $Bu \in D(\mathfrak{A})$, $\mathfrak{A}B \in \mathcal{L}(U,X)$ and
$$\mathfrak{B}Bu = u, \qquad u \in U. \tag{11.5}$$

Part 2 of the definition implies in particular that the range of the operator $\mathfrak{B}$ equals U. Note that part 1 of the definition guarantees that the system (11.3) possesses a unique solution for the choice $u = 0$, i.e., the homogeneous equation is well-posed. Part 2 allows us to choose every value in U for the input $u(t)$ at time $t \geq 0$. In other words, the values of an input are not restricted, which is a plausible condition.

We say that the function $x : [0,\tau] \to X$ is a *classical solution* of the boundary control system of Definition 11.1.1 on $[0,\tau]$ if x is a continuously differentiable function, $x(t) \in D(\mathfrak{A})$ for all $t \in [0,\tau]$, and $x(t)$ satisfies (11.3) for all $t \in [0,\tau]$.

The function $x : [0, \infty) \to X$ is a *classical solution on* $[0, \infty)$ if x is a classical solution on $[0, \tau]$ for every $\tau > 0$.

For a boundary control system, we can apply a similar trick as the one applied to the transport equation. This is the subject of the following theorem. It turns out that $v(t) = x(t) - Bu(t)$ is the solution of the abstract differential equation

$$\dot{v}(t) = Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t), \qquad v(0) = v_0. \tag{11.6}$$

Since A is the infinitesimal generator of a C_0-semigroup and B and $\mathfrak{A}B$ are bounded linear operators, Theorem 10.1.3 implies that equation (11.6) has a unique classical solution for $v_0 \in D(A)$ and $u \in C^2([0, \tau]; U)$. Furthermore, we can prove the following relation between the (classical) solutions of (11.3) and (11.6).

Theorem 11.1.2. *Consider the boundary control system* (11.3) *and the abstract differential equation* (11.6). *Assume that* $u \in C^2([0, \tau]; U)$. *If* $v_0 = x_0 - Bu(0) \in D(A)$, *then the classical solutions of* (11.3) *and* (11.6) *on* $[0, \tau]$ *are related by*

$$v(t) = x(t) - Bu(t). \tag{11.7}$$

Furthermore, the classical solution of (11.3) *is unique.*

Proof. Suppose that v is a classical solution of (11.6). Then for $t \in [0, \tau]$ we get $v(t) \in D(A) \subset D(\mathfrak{A}) \subset D(\mathfrak{B})$, $Bu(t) \in D(\mathfrak{B})$, and

$$\mathfrak{B}x(t) = \mathfrak{B}(v(t) + Bu(t)) = \mathfrak{B}v(t) + \mathfrak{B}Bu(t) = u(t),$$

where we have used that $v(t) \in D(A) \subset \ker \mathfrak{B}$ and that equation (11.5) holds. Furthermore, equation (11.7) implies

$$\begin{aligned}
\dot{x}(t) &= \dot{v}(t) + B\dot{u}(t) \\
&= Av(t) - B\dot{u}(t) + \mathfrak{A}Bu(t) + B\dot{u}(t) && \text{by (11.6)} \\
&= Av(t) + \mathfrak{A}Bu(t) \\
&= \mathfrak{A}(v(t) + Bu(t)) && \text{by (11.4)} \\
&= \mathfrak{A}x(t) && \text{by (11.7).}
\end{aligned}$$

Thus, if v is a classical solution of (11.6), then x defined by (11.7) is a classical solution of (11.3).

The other implication is proved similarly. The uniqueness of the classical solutions of (11.3) follows from the uniqueness of the classical solutions of (11.6), see Theorem 10.1.3. ☐

Remark 11.1.3. In Theorem 11.1.2 we assume that $v_0 = x_0 - Bu(0) \in D(A)$. However, this can equivalently be formulated in a condition on x_0 and $u(0)$. More precisely, the following equivalence is easy to prove, see Exercise 11.1: If (11.3) defines a boundary control system, then the difference $x_0 - Bu(0)$ is an element of the domain of A if and only if $x_0 \in D(\mathfrak{A})$ and $\mathfrak{B}x_0 = u(0)$.

The (mild) solution of (11.6) is given by

$$v(t) = T(t)v_0 + \int_0^t T(t-s)\left(\mathfrak{A}Bu(s) - B\dot{u}(s)\right)\, ds \tag{11.8}$$

for every $v_0 \in X$ and every $u \in H^1([0,\tau];U)$, $\tau > 0$. Therefore, the function

$$x(t) = T(t)(x_0 - Bu(0)) + \int_0^t T(t-s)\left(\mathfrak{A}Bu(s) - B\dot{u}(s)\right)\, ds + Bu(t) \tag{11.9}$$

is called the *mild solution* of the abstract boundary control system (11.3) for every $x_0 \in X$ and every $u \in H^1([0,\tau];U)$ $\tau > 0$.

As an example we study again the controlled transport equation of equation (11.1).

Example 11.1.4. We consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1] \\ x(1,t) &= u(t), && t \geq 0 \end{aligned}$$

for an input $u \in H^1(0,\tau)$. In order to write this example in the form (11.3) we choose $X = L^2(0,1)$ and

$$\begin{aligned} \mathfrak{A}x &= \frac{dx}{d\zeta}, && D(\mathfrak{A}) = H^1(0,1), \\ \mathfrak{B}x &= x(1), && D(\mathfrak{B}) = D(\mathfrak{A}). \end{aligned}$$

These two operators satisfy the assumption of a boundary control system. More precisely: the operators $\mathfrak{A}$ and $\mathfrak{B}$ are linear, $\mathfrak{A}$ restricted to the domain $D(\mathfrak{A}) \cap \ker \mathfrak{B}$ generates a C_0-semigroup, see Exercise 6.3. Furthermore, the range of $\mathfrak{B}$ is $\mathbb{C} = U$ and the choice $B = {}_{[0,1]}$ implies $\mathfrak{B}Bu = u$. Using the fact that $\mathfrak{A}B = 0$, we conclude from equation (11.9) that the mild solution is given by

$$\begin{aligned} x(t) &= T(t)(x_0 - Bu(0)) + \int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))\, ds + Bu(t) \\ &= T(t)(x_0 - u(0)) - \int_0^t T(t-s) \ {}_{[0,1]}\dot{u}(s)\, ds + u(t). \end{aligned}$$

Using the precise representation of the shift-semigroup, see Exercise 6.3, we can write the solution of the boundary controlled partial differential equation as

$$x(\zeta,t) = (x_0(\zeta+t) - u(0)) \ {}_{[0,1]}(\zeta+t) - \int_0^t \ {}_{[0,1]}(\zeta+t-s)\dot{u}(s)\, ds + u(t).$$

If $\zeta + t > 1$, we have

$$x(\zeta, t) = -u(\tau)|_{\zeta+t-1}^{t} + u(t) = u(\zeta + t - 1),$$

and if $\zeta + t \leq 1$, then

$$x(\zeta, t) = x_0(\zeta + t) - u(0) - u(\tau)|_0^t + u(t) = x_0(\zeta + t).$$

Or equivalently,

$$x(\zeta, t) = \begin{cases} x_0(\zeta + t), & \zeta + t \leq 1, \\ u(\zeta + t - 1), & \zeta + t > 1, \end{cases} \tag{11.10}$$

which proves our claim made in the introduction of this chapter.

It is easy to show that this example cannot be written as an abstract control system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{11.11}$$

with B a bounded operator.

If the controlled transport equation were of the form (11.11), then Theorem 10.1.3 would imply that $x(t) \in D(A)$, whenever $x(0) \in D(A)$ and $u \in C^1([0, \tau]; X)$. Choosing $x_0 = 0$, $t = \frac{1}{2}$, and $u = 1$, (11.10) implies that $x(\zeta, \frac{1}{2}) = 0$ if $\zeta \leq \frac{1}{2}$ and $x(\zeta, \frac{1}{2}) = 1$ if $\zeta > \frac{1}{2}$, which is in contradiction to $x(\frac{1}{2}) \in D(A) \subset D(\mathfrak{A}) = H^1(0, 1)$. Summarizing, the boundary controlled transport equation of this example cannot be written in the form (11.11).

The controlled transport equation is a simple example of our general class of port-Hamiltonian systems. This example could be written as a boundary control system. In section 11.3 we show that this holds in general for a port-Hamiltonian system. However, before we do this, we add an output to the boundary control system.

11.2 Outputs for boundary control systems

In the previous section we showed that a system with control at the boundary possesses a (unique) classical solution, provided the input and initial condition are smooth enough. In Section 10.2, we saw that having a solution for the state equation, easily led to the solution of the output equation. In this section, we show that for classical solutions of a boundary control system a similar property holds.

We add a (boundary) *output* to the boundary control system (11.3).

$$\dot{x}(t) = \mathfrak{A}x(t), \quad x(0) = x_0, \tag{11.12}$$

$$\mathfrak{B}x(t) = u(t), \tag{11.13}$$

$$\mathfrak{C}x(t) = y(t) \tag{11.14}$$

where $(\mathfrak{A}, \mathfrak{B})$ satisfies the conditions of a boundary control system, see Definition 11.1.1 and $\mathfrak{C}$ is a linear operator from $D(\mathfrak{A})$ to Y with Y a Hilbert space.

Since a classical solution of (11.12)–(11.13) takes values in the domain of $\mathfrak{A}$, and since $\mathfrak{C}$ is well-defined on this space, there is no difficulty in solving (11.14). We summarize the answer in the following theorem.

Theorem 11.2.1. *Consider the boundary control system* (11.12)–(11.14) *with* $(\mathfrak{A}, \mathfrak{B})$ *satisfying the conditions of Definition* 11.1.1 *and* $\mathfrak{C}$ *a linear operator from* $D(\mathfrak{A})$ *to* Y.

If $u \in C^2([0,\tau]; U)$, $x_0 \in D(\mathfrak{A})$, *and* $\mathfrak{B}x_0 = u(0)$, *then the classical solution of* (11.12)–(11.14) *is given by*

$$x(t) = T(t)(x_0 - Bu(0)) + \int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))\,ds + Bu(t),$$

$$y(t) = \mathfrak{C}T(t)(x_0 - Bu(0)) + \mathfrak{C}\int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))\,ds + \mathfrak{C}Bu(t).$$

If $D(\mathfrak{A})$ *is a Hilbert space, and* $\mathfrak{C}$ *is a bounded linear operator from* $D(\mathfrak{A})$ *to* Y, *then the output* y *is a continuous function.*

For an example of a boundary control system with a boundary output, we refer to Example 11.3.6. This example is a port-Hamiltonian system with boundary control and observation. In the following section we show that these systems fall into the class of boundary control systems.

11.3 Port-Hamiltonian systems as boundary control systems

In this section we add a boundary control to a port-Hamiltonian system and we show that the assumptions of a boundary control system are satisfied. The port-Hamiltonian system with control is given by

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0(\mathcal{H}(\zeta)x(\zeta,t)), \tag{11.15}$$

$$u(t) = W_{B,1}\begin{bmatrix} f_\partial(t), \\ e_\partial(t) \end{bmatrix} \tag{11.16}$$

$$0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}. \tag{11.17}$$

We make the following assumptions.

Assumption 11.3.1.

- $P_1 \in \mathbb{K}^{n\times n}$ is invertible and self-adjoint;
- $\mathcal{H} \in L^\infty([a,b]; \mathbb{K}^{n\times n})$, $\mathcal{H}(\zeta)$ is self-adjoint for a.e. $\zeta \in [a,b]$ and there exist $M, m > 0$ such that $mI \le \mathcal{H}(\zeta) \le MI$ for a.e. $\zeta \in [a,b]$;

- $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \in \mathbb{K}^{n\times 2n}$ has full rank.

Thus, in particular, P_1 and $\mathcal{H}$ satisfy the assumptions of Definition 7.1.2. We recall that the boundary effort and boundary flow are given by, see (7.26)

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = R_0 \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix},$$

where R_0 is the invertible $2n \times 2n$-matrix defined in (7.27).

We can write the port-Hamiltonian system (11.15)–(11.17) as a boundary control system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ \mathfrak{B}x(t) &= u(t), \end{aligned}$$

by defining

$$\mathfrak{A}x = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x) + P_0(\mathcal{H}x), \tag{11.18}$$

$$D(\mathfrak{A}) = \left\{ x \in L^2([a,b];\mathbb{K}^n) \mid \mathcal{H}x \in H^1([a,b];\mathbb{K}^n), W_{B,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \tag{11.19}$$

$$\mathfrak{B}x = W_{B,1} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \tag{11.20}$$

$$D(\mathfrak{B}) = D(\mathfrak{A}). \tag{11.21}$$

Following Chapter 7, we choose the Hilbert space $X = L^2([a,b];\mathbb{K}^n)$ equipped with the inner product

$$\langle f, g \rangle_X := \frac{1}{2} \int_a^b f(\zeta)^* \mathcal{H}(\zeta) g(\zeta)\, d\zeta \tag{11.22}$$

as the state space. The input space U equals $\mathbb{K}^m$, where m is the number of rows of $W_{B,1}$[1]. We are now in the position to show that the controlled port-Hamiltonian system is indeed a boundary control system.

Theorem 11.3.2. *If the operator*

$$Ax = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x) + P_0(\mathcal{H}x) \tag{11.23}$$

with the domain

$$D(A) = \left\{ x \in X \mid \mathcal{H}x \in H^1([a,b];\mathbb{K}^n), \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \right\} \tag{11.24}$$

generates a C_0-semigroup on X, then the system (11.15)–(11.17) *is a boundary control system on X.*

[1] Note that m has two meanings: It is used as a lower-bound for $\mathcal{H}$ and as the dimension of our input space.

Proof. Equations (11.19) and (11.20) imply that $D(A) = D(\mathfrak{A}) \cap \ker \mathfrak{B}$, and hence part 1 of Definition 11.1.1 is satisfied.

The $n \times 2n$-matrix W_B has full rank n and R_0 is an invertible matrix. Thus there exists a $2n \times n$-matrix S such that

$$W_B R_0 S = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} R_0 S = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \tag{11.25}$$

where I_m is the identity matrix on $\mathbb{K}^m$. A possible choice for the matrix S is $S = R_0^{-1} W_B^* (W_B W_B^*)^{-1} \left[\begin{smallmatrix} I_m & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We write $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right]$, where S_{11} and S_{21} are $n \times m$-matrices, and we define the operator $B \in \mathcal{L}(\mathbb{K}^m, X)$ by

$$(Bu)(\zeta) := \mathcal{H}(\zeta)^{-1} \left(S_{11} \frac{\zeta - a}{b - a} + S_{21} \frac{b - \zeta}{b - a} \right) u.$$

The definition of B implies that Bu is a square integrable function and that $\mathcal{H}Bu \in H^1([a,b]; \mathbb{K}^n)$. Furthermore, from (11.25) it follows that $W_{B,2} R_0 \left[\begin{smallmatrix} S_{11} \\ S_{21} \end{smallmatrix}\right] = 0$. Combining this with the definition of the boundary effort and boundary flow, we obtain that $Bu \in D(\mathfrak{A})$. Furthermore, B and $\mathfrak{A}B$ are linear bounded operators from $\mathbb{K}^m$ to X and using (11.25) once more, we obtain

$$\mathfrak{B}Bu = W_{B,1} R_0 \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix} u = u.$$

Thus the port-Hamiltonian system is indeed a boundary control system. □

Remark 11.3.3. An essential condition in the above theorem is that A given by (11.23) with domain (11.24) generates a C_0-semigroup. Theorem 7.2.4 and Assumption 11.3.1 imply that this holds in particular when $P_0^* = -P_0$ and $W_B \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] W_B^* \geq 0$.

Since the term $P_0 \mathcal{H}$ can be seen as a bounded perturbation of (11.18) with $P_0 = 0$, Theorem 7.2.4 and Theorem 10.3.1 show that A given by (11.23) with domain (11.24) generates a C_0-semigroup when $W_B \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] W_B^* \geq 0$.

As an example we once more study the controlled transport equation.

Example 11.3.4. We consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \frac{\partial x}{\partial \zeta}(\zeta, t), \qquad && \zeta \in [0,1],\ t \geq 0 \\ x(\zeta, 0) &= x_0(\zeta), && \zeta \in [0,1]. \end{aligned} \tag{11.26}$$

This system can be written as a port-Hamiltonian system (11.15) by choosing $n = 1$, $P_0 = 0$, $P_1 = 1$ and $\mathcal{H} = 1$. Therefore, we have

$$R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x(1,t) - x(0,t) \\ x(1,t) + x(0,t) \end{bmatrix}.$$

Since $n = 1$, we have the choice of either applying one control or no control at all. Adding a boundary control, the control can be written as, see (11.16),

$$u(t) = \begin{bmatrix} a & b \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} x(1,t) - x(0,t) \\ x(1,t) + x(0,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \left[(a+b)x(1,t) + (b-a)x(0,t) \right]. \tag{11.27}$$

Note that $W_B = [a \; b]$ has full rank if and only if $a^2 + b^2 \neq 0$. We assume this from now on.

By Theorem 11.3.2 we only have to check whether the homogeneous partial differential equation generates a C_0-semigroup. Using Remark 11.3.3 this holds if $2ab \geq 0$. Thus possible boundary controls for the transport equation are

$$\begin{aligned} u(t) &= x(1,t), & (a = b = \frac{\sqrt{2}}{2}), \\ u(t) &= 3x(1,t) - x(0,t), & (a = \sqrt{2}, b = 2\sqrt{2}). \end{aligned}$$

However, this remark does not provide an answer for the control $u(t) = -x(1,t) + 3x(0,t)$. By using other techniques it can be shown that (11.26) with this control is also a boundary control system, see Chapter 13.

Next we focus on a boundary observation for port-Hamiltonian systems. We develop conditions on the boundary observation guaranteeing that a certain balance equation is satisfied, which is important in Chapters 12 and 13. The standard Hamiltonian system with boundary control and boundary observation is given by

$$\dot{x}(t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x(t)) + P_0(\mathcal{H}x(t)), \tag{11.28}$$

$$u(t) = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{11.29}$$

$$y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}. \tag{11.30}$$

It is assumed that $P_1, \mathcal{H}$ and W_B satisfy the conditions of Assumption 11.3.1. Note that we have taken $W_B = W_{B,1}$ or equivalently $W_{B,2} = 0$. In other words, we are using the maximal number of controls.

The output equation is formulated very similar to the control equation. So we assume that the output space $Y = \mathbb{K}^k$, and thus W_C is a matrix of size $k \times 2n$. Since we want the outputs to be independent, we assume that W_C has full rank. Furthermore, since we do not want to measure quantities that we already have chosen as an input, see (11.29), we assume that the matrix $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ has full rank.

If we have full measurements, i.e., $k = n$, then the above assumptions imply that the matrix $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is invertible. Combining this assumption with the fact that $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ is invertible, we obtain that the product $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right] \Sigma \left[\begin{smallmatrix} W_B^* & W_C^* \end{smallmatrix}\right]$ is invertible as well. Its inverse is defined as

$$P_{W_B,W_C} := \left(\begin{bmatrix} W_B \\ W_C \end{bmatrix} \Sigma \begin{bmatrix} W_B^* & W_C^* \end{bmatrix} \right)^{-1} = \begin{bmatrix} W_B \Sigma W_B^* & W_B \Sigma W_C^* \\ W_C \Sigma W_B^* & W_C \Sigma W_C^* \end{bmatrix}^{-1}. \tag{11.31}$$

We choose the same state space as in Section 11.3, i.e, $X = L^2([a,b];\mathbb{K}^n)$ with inner product (11.22).

Theorem 11.3.5. *Consider the system* (11.28)–(11.30)*, satisfying Assumption* 11.3.1*,* $W_C \in \mathbb{K}^{k\times 2n}$ *and* $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right] \in \mathbb{K}^{(k+n)\times 2n}$ *having full rank.*

Assume that the operator A *defined by* (11.23) *and* (11.24) *generates a* C_0*-semigroup on* X*. Then for every* $u \in C^2([0,\infty);\mathbb{K}^n)$*,* $\mathcal{H}x(0) \in H^1([a,b];\mathbb{K}^n)$*, and* $u(0) = W_B \begin{bmatrix} f_\partial(0) \\ e_\partial(0) \end{bmatrix}$*, the system* (11.28)–(11.30) *has a unique (classical) solution, with* $\mathcal{H}x(t) \in H^1([a,b];\mathbb{K}^n)$*,* $t \geq 0$*, and the output* y *is continuous.*

Furthermore, if additionally $P_0^* = -P_0$ *and* $k = n$*, then the following balance equation is satisfied for every* $t \geq 0$*:*

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2}\begin{bmatrix} u^*(t) & y^*(t) \end{bmatrix} P_{W_B,W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}. \tag{11.32}$$

Proof. Theorem 11.3.2 implies that the system (11.28)–(11.29) is a boundary control system on X, where the operators $\mathfrak{A}$ and $\mathfrak{B}$ are given by

$$\mathfrak{A}x = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x) + P_0(\mathcal{H}x), \qquad \mathfrak{B}x = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix},$$
$$D(\mathfrak{A}) = D(\mathfrak{B}) = \left\{ x \in L^2([a,b];\mathbb{K}^n) \mid \mathcal{H}x \in H^1([a,b];\mathbb{K}^n) \right\}.$$

Let A and B be the corresponding operators satisfying the properties of Definition 11.1.1. By assumption we have $u \in C^2([0,\infty);\mathbb{K}^n)$, $x(0) \in D(\mathfrak{A})$, and $\mathfrak{B}x(0) = u(0)$. Thus Remark 11.1.3 and Theorem 11.1.2 imply that (11.28)–(11.29) possesses a unique classical solution with $\mathcal{H}x(t) \in H^1([a,b];\mathbb{K}^n)$, $t \geq 0$. Combining equations (11.30) and (11.14), we see that

$$\mathfrak{C}x = W_C R_0 \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix}.$$

This is clearly a linear bounded operator from the domain of $\mathfrak{A}$ to $\mathbb{C}^k$, and thus Theorem 11.2.1 implies that the output corresponding to a classical solution is a continuous function.

If $P_0 = -P_0^*$ and $k = n$, then we may apply Theorem 7.1.5. Using this theorem and the fact that $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$, the classical solution x satisfies

$$\begin{aligned}
\frac{d}{dt}\|x(t)\|_X^2 &= \frac{1}{2}(f_\partial^*(t)e_\partial(t) + e_\partial^*(t)f_\partial(t)) \\
&= \frac{1}{2}\begin{bmatrix} f_\partial^*(t) & e_\partial^*(t) \end{bmatrix} \Sigma \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} u^*(t) & y^*(t) \end{bmatrix} \begin{bmatrix} W_B^* & W_C^* \end{bmatrix}^{-1} \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-1} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} u^*(t) & y^*(t) \end{bmatrix} P_{W_B,W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix},
\end{aligned}$$

which completes the proof. □

As an example we consider the vibrating string.

Example 11.3.6. We consider the vibrating string on the spatial interval $[a, b]$ for

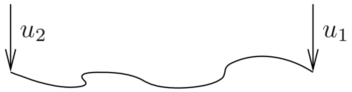

Figure 11.1: The vibrating string with two controls

which the forces at both ends are the control variables, and the velocities at these ends are the outputs. In Example 7.1.1 we saw that the model is given by

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta, t)\right), \tag{11.33}$$

where $\zeta \in [a, b]$ is the spatial variable, $w(\zeta, t)$ is the vertical position of the string at position ζ and time t, T is the Young's modulus of the string, and ρ is the mass density. This system has the energy/Hamiltonian

$$E(t) = \frac{1}{2}\int_a^b \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta, t)\right)^2 + T(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta, t)\right)^2 d\zeta. \tag{11.34}$$

Furthermore, it is a port-Hamiltonian system with state $x = \left[\begin{smallmatrix} \rho \frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{smallmatrix}\right]$, $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $P_0 = 0$, and $\mathcal{H}(\zeta) = \left[\begin{smallmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{smallmatrix}\right]$. The boundary effort and boundary flow for the vibrating string are given by, see Example 7.2.5,

$$f_\partial = \frac{1}{\sqrt{2}}\left[\begin{array}{c} T(b)\frac{\partial w}{\partial \zeta}(b) - T(a)\frac{\partial w}{\partial \zeta}(a) \\ \frac{\partial w}{\partial t}(b) - \frac{\partial w}{\partial t}(a) \end{array}\right], \; e_\partial = \frac{1}{\sqrt{2}}\left[\begin{array}{c} \frac{\partial w}{\partial t}(b) + \frac{\partial w}{\partial t}(a) \\ T(b)\frac{\partial w}{\partial \zeta}(b) + T(a)\frac{\partial w}{\partial \zeta}(a) \end{array}\right]. \tag{11.35}$$

As depicted in Figure 11.1, the forces at the ends are the inputs and the velocity at both ends are the outputs, i.e.,

$$u(t) = \left[\begin{array}{c} T(b)\frac{\partial w}{\partial \zeta}(b, t) \\ T(a)\frac{\partial w}{\partial \zeta}(a, t) \end{array}\right], \qquad y(t) = \left[\begin{array}{c} \frac{\partial w}{\partial t}(b, t) \\ \frac{\partial w}{\partial t}(a, t) \end{array}\right]. \tag{11.36}$$

Using (11.35) we find that W_B and W_C are given as

$$W_B = \frac{1}{\sqrt{2}}\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{array}\right], \qquad W_C = \frac{1}{\sqrt{2}}\left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]. \tag{11.37}$$

W_B and W_C are 2×4-matrices of full rank and $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is invertible. Furthermore,

$$W_B \Sigma W_B^* = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right].$$

By Theorem 7.2.4 we conclude that the associated A generates a contraction semigroup on the energy space. In fact, it generates a unitary group, see Exercise 7.2. Hence our system satisfies the conditions of Theorem 11.3.5. In particular, it is a boundary control system, and since $P_0 = 0$ the balance equation (11.32) holds. Since

$$W_C \Sigma W_C^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad W_C \Sigma W_B^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we find that

$$\frac{d}{dt}\|x(t)\|_X^2 = \frac{1}{2}\left(u_1^*(t)y_1(t) - u_2^*(t)y_2(t) + y_1^*(t)u_1(t) - y_2^*(t)u_2(t)\right). \qquad (11.38)$$

Note that this equals equation (7.3).

11.4 Exercises

11.1. Prove the assertion in Remark 11.1.3

11.2. Consider the transmission line of Exercise 7.1 for any of the following boundary condition:

(a) At the left-end the voltage equal $u_1(t)$ and at the right-end the voltage equals the input $u_2(t)$.

(b) At the left-end we put the voltage equal to zero and at the right-end the voltage equals the input $u(t)$.

(c) At the left-end we put the voltage equal to $u(t)$ and at the right-end the voltage equals R times the current, for some $R > 0$.

Show that these systems can be written as a boundary control system.

11.3. Consider the vibrating string of Example 7.1.1 with the boundary conditions

$$\frac{\partial w}{\partial t}(a,t) = 0 \quad \text{and} \quad T(b)\frac{\partial w}{\partial \zeta}(b,t) = u(t) \quad t \geq 0.$$

(a) Reformulate this system as a boundary control system.

(b) Prove that we have a well-defined input-output system, if we measure the velocity at the right-hand side, $y(t) = \frac{\partial w}{\partial t}(b,t)$.

(c) Does a balance equation like (11.32) hold?

11.4. In the formulation of port-Hamiltonian systems as boundary control systems, we have the possibility that some boundary conditions are set to zero, see (11.17). However, when we add an output, this possibility was excluded, see (11.28)–(11.30). In this exercise we show that this did not pose a restriction to the theory.

(a) Show that if $W_{B,2} = 0$, i.e., $W_B = W_{B,1}$, and $W_B \Sigma W_B^* \geq 0$, then (11.15) with control (11.16) is a well-defined boundary control system. What is the domain of the infinitesimal generator A?

(b) Let W_B be a $n \times 2n$ matrix of full rank satisfying $W_B \Sigma W_B^* \geq 0$. We decompose $u = W_B \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$ as

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}.$$

Show that the choice $u_2 = 0$ is allowed. Furthermore, show that it leads to the same boundary control system as in (11.15)–(11.17).

11.5. Consider the coupled strings of Exercise 7.4. Now we apply a force $u(t)$, to the bar in the middle, see Figure 11.2. This implies that the force balance in

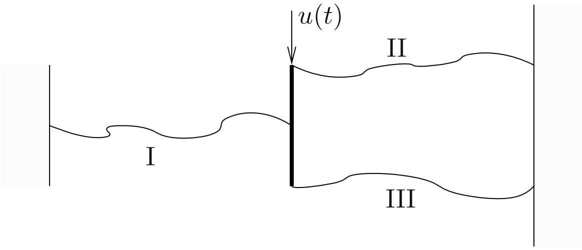

Figure 11.2: Coupled vibrating strings with external force

the middle reads

$$T_{\mathrm{I}}(b)\frac{\partial w_{\mathrm{I}}}{\partial \zeta}(b) = T_{\mathrm{II}}(a)\frac{\partial w_{\mathrm{II}}}{\partial \zeta}(a) + T_{\mathrm{III}}(a)\frac{\partial w_{\mathrm{III}}}{\partial \zeta}(a) + u(t).$$

(a) Formulate the coupled vibrating strings with external force as a boundary control system.

(b) Additionally, we measure the velocity of the bar in the middle. Reformulate the system with this choice for the output as a system of form (11.28)–(11.30).

(c) For the input and output defined above, determine the power balance in terms of the input and output, see (11.32).

11.5 Notes and references

The boundary control system as presented in this chapter was introduced by Fattorini [16], and has become a standard way of reformulating partial differential equations with boundary control as an abstract system. The application to port-Hamiltonian systems of Sections 11.3 and 11.2 can be found in [36].

Chapter 12

Transfer Functions

In this chapter we introduce the concept of transfer functions. In the system and control literature a transfer function is usually defined via the Laplace transform. However, in this chapter we use a different approach. Nevertheless, we explain the Laplace transform-approach by means of an example, that is, we consider the ordinary differential equation

$$\ddot{y}(t) + 3\dot{y}(t) - 7y(t) = -\dot{u}(t) + 2u(t), \tag{12.1}$$

where the dot denotes the derivative with respect to time.

Let $\mathfrak{L}$ denote the Laplace transform, and let F be the Laplace transform of the function f, i.e., $(\mathfrak{L}(f))(s) = F(s)$. Recall that the following rules hold for the Laplace transform

$$\begin{aligned} \left(\mathfrak{L}(\dot{f})\right)(s) &= sF(s) - f(0), \\ \left(\mathfrak{L}(\ddot{f})\right)(s) &= s^2F(s) - sf(0) - \dot{f}(0). \end{aligned}$$

Assuming that $y(0) = \dot{y}(0) = u(0) = 0$ and applying the Laplace transform to the differential equation (12.1), we obtain the algebraic equation

$$s^2Y(s) + 3sY(s) - 7Y(s) = -sU(s) + 2U(s). \tag{12.2}$$

Thus we get the following relation between the Laplace transform of u and y:

$$Y(s) = \frac{-s+2}{s^2+3s-7}U(s). \tag{12.3}$$

The rational function $\frac{-s+2}{s^2+3s-7}$ is called the transfer function associated to the differential equation (12.1).

This is the standard approach to transfer functions. However, this approach faces some difficulties, especially when we want to extend the concept to partial

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differential equations. One of the difficulties is that the functions u and y have to be Laplace transformable. Considering u as input or control and y as output, the assumption that u is Laplace transformable is not very restrictive. However, once u is chosen, y is given by the differential equation, and it is a priori not known whether y is Laplace transformable. Another difficulty is that the Laplace transform of a function only exists in some right half-plane of the complex plane, which implies that equality (12.3) only holds for those s in the right-half plane for which the Laplace transform of u and y both exist. The right-half plane in which the Laplace transform of a function exists is named the region of convergence. Even for the simple differential equation (12.1) equality (12.3) does not hold everywhere. Taking into account the region of convergence of both u and y, equation (12.3) can only hold for those s which lie right of the zeros of the polynomial s^2+3s-7. However, for applications it is important to know the transfer function on a larger domain in the complex plane. For the finite-dimensional system (12.1) the transfer function G is given by $G(s) = \frac{-s+2}{s^2+3s-7}$ for all $s \in \mathbb{C}$, where s not a zero of s^2+3s-7.

To overcome all these difficulties and to justify $G(s) = \frac{-s+2}{s^2+3s-7}$ on a larger domain in the complex plane, we define the transfer function in a different way. We try to find solutions of the differential equation which are given as exponential functions. Again we illustrate this approach for the simple differential equation (12.1). Given $s \in \mathbb{C}$, we try to find a solution pair of the form $(u, y) = (e^{st}, y_s e^{st})_{t\geq 0}$. If for an s such a solution exists, and it is unique, then we call y_s the transfer function of (12.1) in the point s. Substituting this solution pair into the differential equation, we obtain

$$s^2 y_s e^{st} + 3s y_s e^{st} - 7 y_s e^{st} = -s e^{st} + 2e^{st}, \qquad t \geq 0. \tag{12.4}$$

We recognize the common term e^{st} which is never zero, and hence (12.4) is equivalent to

$$s^2 y_s + 3s y_s - 7 y_s = -s + 2. \tag{12.5}$$

This is uniquely solvable for y_s if and only if $s^2 + 3s - 7 \neq 0$. Furthermore, y_s is given by $y_s = \frac{-s+2}{s^2+3s-7}$.

Hence it is possible to define the transfer function without running into mathematical difficulties. The same approach works well for p.d.e.'s and abstract differential equations as the concept of a solution is well-defined.

12.1 Basic definition and properties

In this section we start with a very general definition of a transfer function, which even applies to systems not described by a p.d.e, but via e.g. a difference differential equation or an integral equation. Therefore, we first introduce the notion of a general system. Let $\mathbb{T} := [0, \infty)$ be the time axis. Furthermore, we distinguish three spaces, U, Y, and R. U and Y are the input- and output space, respectively, whereas R contains the remaining variables. In our examples, R will become the state space X. Since $s \in \mathbb{C}$, the exponential solutions will be complex-valued.

Therefore we assume that U, R and Y are complex Hilbert spaces. A *system* $\mathfrak{S}$ is a subset of $L^1_{\text{loc}}([0,\infty); U \times R \times Y)$, i.e., a subset of all locally integrable functions from the time axis $\mathbb{T}$ to $U \times R \times Y$. Note that two functions f and g are equal in $L^1_{\text{loc}}([0,\infty); U \times R \times Y)$ if $f(t) = g(t)$ for almost every $t \geq 0$.

Definition 12.1.1. Let $\mathfrak{S}$ be a system, s be an element of $\mathbb{C}$, and $u_0 \in U$. We say that $(u_0 e^{st}, r(t), y(t))_{t\geq 0}$ is an *exponential solution* in $\mathfrak{S}$ if there exist $r_0 \in R$, $y_0 \in Y$, such that $(u_0 e^{st}, r_0 e^{st}, y_0 e^{st}) = (u_0 e^{st}, r(t), y(t))$ for a.e. $t \geq 0$.

Let $s \in \mathbb{C}$. If for every $u_0 \in U$ there exists an exponential solution, and the corresponding output trajectory $y_0 e^{st}, t \in [0,\infty)$ is unique, then we call the mapping $u_0 \mapsto y_0$ the *transfer function at* s. We denote this mapping by $G(s)$. Let $\Omega \subset \mathbb{C}$ be the set consisting of all s for which the transfer function at s exists. The mapping $s \in \Omega \mapsto G(s)$ is defined as the *transfer function* of the system $\mathfrak{S}$.

In our applications r will usually be the state x. We begin by showing that the transfer function for the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{12.6}$$

$$y(t) = Cx(t) + Du(t), \tag{12.7}$$

see (10.16) and (10.17), where B, C and D are bounded operators and A generates a strongly continuous semigroup, exists and is given by the formula $G(s) = C(sI - A)^{-1}B + D$, for s in the resolvent set, $\rho(A)$, of A.

Theorem 12.1.2. *Consider the linear system* (12.6)–(12.7)*, with B, C, and D bounded operators. The solution of the system is given by the mild solution, see Theorem* 10.2.1.

If $(u(t), x(t), y(t))_{t\geq 0}$ is an exponential solution of (12.6)–(12.7)*, then x is a classical solution of* (12.6)*. Furthermore, for $s \in \rho(A)$, the transfer function exists and is given by*

$$G(s) = C(sI - A)^{-1}B + D. \tag{12.8}$$

Proof. The mild solution of (12.6) with initial condition $x(0) = x_0$ is uniquely determined and given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau. \tag{12.9}$$

For an exponential solution this equation should equal

$$x_0 e^{st} = T(t)x_0 + \int_0^t T(t-\tau)Bu_0 e^{s\tau}d\tau. \tag{12.10}$$

Taking $x_0 = (sI - A)^{-1}Bu_0$, the right-hand side of this equation can be written as

$$T(t)x_0 + \int_0^t T(t-\tau)(sI - A)x_0 e^{s\tau}d\tau$$
$$= T(t)x_0 + e^{st}\int_0^t T(t-\tau)e^{-s(t-\tau)}(sI - A)x_0 d\tau.$$

By Exercise 5.3 the infinitesimal generator of the C_0-semigroup $(T(t)e^{-st})_{t\geq 0}$ is given by $A - sI$. Applying Theorem 5.2.2.4 to the above equation we find that

$$\begin{aligned} T(t)x_0 + e^{st}\int_0^t T(t-\tau)e^{-s(t-\tau)}(sI-A)x_0 d\tau \\ = T(t)x_0 - e^{st}\left(T(t)e^{-st}x_0 - x_0\right) = e^{st}x_0. \end{aligned} \tag{12.11}$$

Thus by choosing $x_0 = (sI - A)^{-1}Bu_0$, we obtain an exponential solution. We show next that this solution is unique. If there are two exponential solutions for the same u_0 and s, then by (12.10) we see that their difference satisfies

$$\tilde{x}_0 e^{st} = T(t)\tilde{x}_0, \qquad t \geq 0, \tag{12.12}$$

for some $\tilde{x}_0 \in X$. Thus $\tilde{x}_0$ is an eigenvector of $T(t)$ for all $t \geq 0$. Since the left-hand side is differentiable so is the right-hand side. Thus $\tilde{x}_0 \in D(A)$ and $A\tilde{x}_0 = s\tilde{x}_0$, see also Exercise 8.5. Since $s \in \rho(A)$, this implies that $\tilde{x}_0 = 0$. Thus the exponential solution is unique.

The output equation of the system yields

$$y_0 e^{st} = y(t) = Cx(t) + Du(t) = Cx_0 e^{st} + Du_0 e^{st} = C(sI-A)^{-1}Bu_0 e^{st} + Du_0 e^{st}.$$

Thus for every $s \in \rho(A)$ the output trajectory corresponding to an exponential solution is uniquely determined, and $y_0 = C(sI - A)^{-1}Bu_0 + Du_0$. This implies that the system possesses a transfer function on $\Omega = \rho(A)$ and the transfer function is given by (12.8). □

Theorem 12.1.2 shows that for linear systems of the form (12.6)–(12.7) the transfer function exists, and is given by $G(s) = C(sI-A)^{-1}B + D$. Unfortunately, port-Hamiltonian systems can in general not be written in the form (12.6)–(12.7) with bounded operators B, C and D. However, by Theorem 11.3.2 every port-Hamiltonian system is a boundary control system. In order to calculate the transfer function of port-Hamiltonian systems, we first focus on transfer functions of boundary control systems. Again we show that every exponential solution is a classical solution.

Theorem 12.1.3. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \qquad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t), \end{aligned} \tag{12.13}$$

where $(\mathfrak{A}, \mathfrak{B})$ *satisfies the conditions of a boundary control system, see Definition 11.1.1 and* $\mathfrak{C}$ *is a linear operator from* $D(\mathfrak{A})$ *to* Y*, where* Y *is a Hilbert space. The solution of the state differential equation is given by the mild solution, see equation (11.9).*

Every exponential solution of (12.13) *is also a classical solution. Furthermore, for $s \in \rho(A)$, the transfer function exists and is given by*

$$G(s) = \mathfrak{C}(sI - A)^{-1} (\mathfrak{A}B - sB) + \mathfrak{C}B. \tag{12.14}$$

For $s \in \rho(A)$ and $u_0 \in U$, $G(s)u_0$ can also be calculated as the (unique) solution of

$$\begin{aligned} sx_0 &= \mathfrak{A}x_0, \\ u_0 &= \mathfrak{B}x_0, \\ G(s)u_0 &= \mathfrak{C}x_0, \end{aligned} \tag{12.15}$$

with $x_0 \in D(\mathfrak{A})$.

Proof. By (11.9) the mild solution of (12.13) is given by

$$x(t) = T(t)(x_0 - Bu(0)) + \int_0^t T(t-\tau) (\mathfrak{A}Bu(\tau) - B\dot{u}(\tau))\, d\tau + Bu(t).$$

Assuming that $(u(t), x(t), y(t))_{t\geq 0}$ is an exponential solution, the above equation becomes

$$e^{st}x_0 = T(t)(x_0 - Bu_0) + \int_0^t T(t-\tau) (\mathfrak{A}Be^{s\tau}u_0 - Bse^{s\tau}u_0)\, d\tau + Be^{st}u_0. \tag{12.16}$$

Applying the Laplace transform, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \max\{\mathrm{Re}(s), \omega_0\}$, where ω_0 is the growth bound of the semigroup $(T(t))_{t\geq 0}$, we obtain

$$\frac{x_0}{\lambda - s} = (\lambda I - A)^{-1}(x_0 - Bu_0) + (\lambda I - A)^{-1} \left(\mathfrak{A}B\frac{u_0}{\lambda - s} - B\frac{su_0}{\lambda - s} \right) + B\frac{u_0}{\lambda - s},$$

or equivalently,

$$\frac{x_0 - Bu_0}{\lambda - s} = (\lambda I - A)^{-1}(x_0 - Bu_0) + (\lambda I - A)^{-1} \left(\mathfrak{A}B\frac{u_0}{\lambda - s} - B\frac{su_0}{\lambda - s} \right).$$

This implies that $x_0 - Bu_0 \in D(A)$, and

$$(\lambda I - A)(x_0 - Bu_0) = (x_0 - Bu_0)(\lambda - s) + \mathfrak{A}Bu_0 - Bsu_0.$$

Subtracting the term $\lambda(x_0 - Bu_0)$ from both sides, we obtain

$$(sI - A)(x_0 - Bu_0) = \mathfrak{A}Bu_0 - Bsu_0, \tag{12.17}$$

or equivalently, for $s \in \rho(A)$, x_0 is uniquely determined as

$$x_0 = (sI - A)^{-1} (\mathfrak{A}B - sB)\, u_0 + Bu_0. \tag{12.18}$$

Hence this shows that if we have for $s \in \rho(A)$ an exponential solution $(u(t), x(t))_{t\geq 0} = (u_0 e^{st}, x_0 e^{st})_{t\geq 0}$, then the x_0 is uniquely determined by u_0. Starting with x_0 defined by (12.18) and reading backwards, we see from equation (12.16) that $x(t) = x_0 e^{st}$ is the solution for $u(t) = u_0 e^{st}$.

By (12.18) we have that $x_0 - Bu_0 \in D(A)$ and since $u \in C^2([0,\infty); U)$, Theorem 11.1.2 implies that x is a classical solution of (12.13). In particular, for all $t \geq 0$, $x(t) \in D(\mathfrak{A})$. By assumption the domain of $\mathfrak{C}$ contains the domain of $\mathfrak{A}$. Hence $y_0 e^{st} = y(t) = \mathfrak{C}x(t)$, holds point-wise in t. The choice $t = 0$ implies $y_0 = \mathfrak{C}x_0$. Combining this equality with (12.18), we obtain that the transfer function is well-defined and is given by (12.14).

Since $x(t) = x_0 e^{st}$ is the classical solution corresponding to the input $u(t) = u_0 e^{st}$ and the initial condition $x(0) = x_0$, the differential equation (12.13) reads

$$\begin{aligned} s x_0 e^{st} &= \mathfrak{A} x_0 e^{st}, \\ u_0 e^{st} &= \mathfrak{B} x_0 e^{st}, \\ y_0 e^{st} &= \mathfrak{C} x_0 e^{st}, \end{aligned}$$

which is equivalent to equation (12.15). The uniqueness of y_0 follows from the uniqueness of x_0, see (12.18). □

We close this section by calculating the transfer function of a controlled transport equation.

Example 12.1.4. Consider the system

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial x}{\partial \zeta}(\zeta,t), && \zeta \in [0,1],\ t \geq 0, \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [0,1], \\ u(t) &= x(1,t), && t \geq 0, \\ y(t) &= x(0,t), && t \geq 0. \end{aligned} \tag{12.19}$$

If we define $\mathfrak{C}x = x(0)$, then it is easy to see that all assumptions of Theorem 12.1.3 are satisfied, see Theorem 11.3.5. Hence we can calculate the transfer function using equation (12.15). For the system (12.19), equation (12.15) reads

$$\begin{aligned} s x_0(\zeta) &= \frac{dx_0}{d\zeta}(\zeta), \\ u_0 &= x_0(1), \\ G(s)u_0 &= x_0(0). \end{aligned}$$

The solution of this differential equation is given by $x_0(\zeta) = \alpha e^{s\zeta}$. Using the other two equations, we see that $G(s) = e^{-s}$.

12.2 Transfer functions for port-Hamiltonian systems

In this section we apply the results of the previous section to the class of port-Hamiltonian systems. Due to the fact that every port-Hamiltonian system is a boundary control system provided the corresponding operator A generates a C_0-semigroup, it is a direct application of Theorem 12.1.3. Recall that a port-Hamiltonian system with boundary control and boundary observation is given by

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}\left(\mathcal{H}(\zeta)x(\zeta,t)\right) + P_0(\mathcal{H}(\zeta)x(\zeta,t)), \tag{12.20}$$

$$u(t) = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{12.21}$$

$$y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{12.22}$$

see (11.28)–(11.30). It is assumed that (12.20)–(12.21) satisfy the assumption of a port-Hamiltonian system, see Section 11.3. Further we assume that $W_B \Sigma W_B^T \geq 0$, where $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$. This guarantees that (12.20)–(12.21) is a boundary control system, see Remark 11.3.3.

We assume we have k (independent) measurements, i.e., W_C is a full rank matrix of size $k \times 2n$ and $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is of full rank.

Theorem 12.2.1. *The transfer function $G(s)$ of the system* (12.20)–(12.22) *is uniquely determined by*

$$s x_0 = P_1 \frac{d}{d\zeta}(\mathcal{H}x_0) + P_0(\mathcal{H}x_0), \tag{12.23}$$

$$u_0 = W_B \begin{bmatrix} f_{\partial,0} \\ e_{\partial,0} \end{bmatrix}, \tag{12.24}$$

$$G(s)u_0 = W_C \begin{bmatrix} f_{\partial,0} \\ e_{\partial,0} \end{bmatrix}, \tag{12.25}$$

where

$$\begin{bmatrix} f_{\partial,0} \\ e_{\partial,0} \end{bmatrix} = R_0 \begin{bmatrix} (\mathcal{H}x_0)(b) \\ (\mathcal{H}x_0)(a) \end{bmatrix}. \tag{12.26}$$

This transfer function has the following properties:

1. *If P_0, P_1, W_B, and W_C are real matrices and $\mathcal{H}$ is real-valued, then $G(s)$ is a real matrix for real s.*

2. *If we have full measurements, i.e., $k = n$, then the transfer function satisfies the equality*

$$Re\,(s)\|x_0\|^2 = \frac{1}{2}\begin{bmatrix} u_0^* & u_0^* G(s)^* \end{bmatrix} P_{W_B,W_C} \begin{bmatrix} u_0 \\ G(s)u_0 \end{bmatrix} \tag{12.27}$$

where P_{W_B,W_C} is the inverse of $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right] \Sigma \left[\begin{smallmatrix} W_B^* & W_C^* \end{smallmatrix}\right]$.

Proof. The proof is a direct combination of Theorems 11.3.5 and 12.1.3. The first theorem implies that the system (12.20)–(12.22) is a well-defined boundary control system and that the output equation is well-defined in the domain of the system operator $\mathfrak{A}$. Hence all conditions of Theorem 12.1.3 are satisfied, and equation (12.15) for the port-Hamiltonian system reads (12.23)–(12.25).

1. To prove the assertion in part 1, we need some notation. We denote by $\overline{v}$ the element-wise complex conjugate of the vector v. Similarly, for the matrix Q. With this notation we find from equation (12.23) that

$$s\overline{x_0} = \overline{sx_0} = \overline{P_1 \frac{d}{d\zeta}(\mathcal{H}x_0)} + \overline{P_0(\mathcal{H}x_0)} = P_1 \frac{d}{d\zeta}(\mathcal{H}\overline{x_0}) + P_0(\mathcal{H}\overline{x_0}),$$

where we have used that s, P_1, P_0 and $\mathcal{H}$ are real-valued. Similarly, we find

$$\overline{u_0} = W_B \begin{bmatrix} \overline{f_{\partial,0}} \\ \overline{e_{\partial,0}} \end{bmatrix},$$

$$\overline{G(s)}\overline{u_0} = W_C \begin{bmatrix} \overline{f_{\partial,0}} \\ \overline{e_{\partial,0}} \end{bmatrix},$$

where

$$\begin{bmatrix} \overline{f_{\partial,0}} \\ \overline{e_{\partial,0}} \end{bmatrix} = R_0 \begin{bmatrix} (\mathcal{H}\overline{x_0})(b) \\ (\mathcal{H}\overline{x_0})(a) \end{bmatrix}.$$

Thus $\overline{G(s)}$ satisfies the equations (12.23)–(12.26) with u_0 replaced $\overline{u_0}$. However, u_0 is arbitrary, and so we see that $\overline{G(s)}$ is also the transfer function at s. Since the transfer function is unique, we conclude that $\overline{G(s)} = G(s)$ and thus G is real-valued.

2. The transfer function is by definition related to the exponential solution

$$(u_0 e^{st}, x_0 e^{st}, \quad G(s)u_0 e^{st})_{t \geq 0}.$$

Hence if we have full measurement, then we may substitute this solution in (11.32). Dividing by $e^{2\mathrm{Re}(s)t}$ gives (12.27). □

By (12.23)–(12.25), the calculation of the transfer function is equivalent to solving an ordinary differential equation. If $\mathcal{H}$ is constant, i.e., independent of ζ, this is relatively easy. However, in general it can be very hard to solve this ordinary differential equation analytically, see Exercise 12.2.

In Theorem 12.2.1 we assumed that there are exactly n controls. However, this does in general not hold and the transfer function can also be calculated if some of the boundary conditions are set to zero. There are two possibilities to calculate the transfer function in this more general situation. Exercise 11.4 shows that this more general system satisfies all the conditions of Theorem 12.1.3, and hence Theorem 12.1.3 can be used to obtain the differential equation determining the transfer function. Another approach is to regard the zero boundary conditions as additional inputs, and to add extra measurements such that we have n controls and n measurements. The requested transfer function is now a sub-block of the $n \times n$-transfer function. We explain this in more detail by means of an example, see Example 12.2.3. Note that (12.27) equals the balance equation (7.16).

Example 12.2.2. Consider the following partial differential equation

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \frac{\partial \mathfrak{h} x}{\partial \zeta}(\zeta,t), && \zeta \in [a,b],\ t \geq 0, \\ x(\zeta,0) &= x_0(\zeta), && \zeta \in [a,b], \\ u(t) &= \mathfrak{h}(b)x(b,t), && t \geq 0, \\ y(t) &= \mathfrak{h}(a)x(a,t), && t \geq 0, \end{aligned} \tag{12.28}$$

where $\mathfrak{h} : [a,b] \to \mathbb{R}$ is a (strictly) positive continuous function. It is easy to see that this is a port-Hamiltonian system with $P_1 = 1$, $P_0 = 0$, and $\mathcal{H} = \mathfrak{h}$. The energy balance equation (7.16) becomes

$$\frac{d}{dt}\int_a^b x(\zeta,t)^* \mathfrak{h}(\zeta) x(\zeta,t)\, d\zeta = |\mathfrak{h}(b)x(b,t)|^2 - |\mathfrak{h}(a)x(a,t)|^2. \tag{12.29}$$

We derive the transfer function by applying Theorem 12.2.1 to the partial differential equation (12.28). Thus we obtain the following ordinary differential equation

$$\begin{aligned} s x_0(\zeta) &= \frac{d\mathfrak{h} x_0}{d\zeta}(\zeta), && \zeta \in [a,b], \\ u_0 &= \mathfrak{h}(b)x_0(b), \\ G(s)u_0 &= \mathfrak{h}(a)x_0(a). \end{aligned} \tag{12.30}$$

The solution of this differential equation is given by

$$x_0(\zeta) = \frac{c_0}{\mathfrak{h}(\zeta)} \exp(p(\zeta)s),$$

where c_0 is a constant not depending on ζ, and

$$p(\zeta) = \int_a^\zeta \mathfrak{h}(\alpha)^{-1} d\alpha.$$

The second equation of (12.30) can be used to calculate c_0. Using the third equation of (12.30) we obtain

$$G(s) = e^{-p(b)s}. \tag{12.31}$$

The strict positivity of $\mathfrak{h}$ on $[a,b]$ implies that $|G(s)| \leq 1$ for all s with non-negative real part. This result also follows from the balance equation (12.29) without calculating the transfer function as follows. Since every exponential solution is a classical solution, for the exponential solution $(u_0 e^{st}, x_0 e^{st}, G(s)u_0 e^{st})_{t\geq 0}$ we obtain

$$\frac{d}{dt}\int_a^b |e^{st}|^2 x_0(\zeta)^* \mathfrak{h}(\zeta) x_0(\zeta)\, d\zeta = |u_0 e^{st}|^2 - |G(s)u_0 e^{st}|^2. \tag{12.32}$$

Using the fact that $|e^{st}|^2 = e^{2\mathrm{Re}(s)t}$, equation (12.29) implies that for all u_0 and all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$ we have that $|G(s)u_0|^2 \leq |u_0|^2$. Since u_0 is a scalar, we conclude that the absolute value of $G(s)$ is bounded by 1 for $\mathrm{Re}(s) \geq 0$.

Example 12.2.3. Consider the vibrating string of Example 11.3.6, as depicted in Figure 12.1. Again we assume that we control the forces at both ends, and mea-

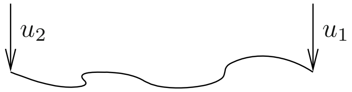

Figure 12.1: The vibrating string with two controls

sure the velocities at the same points. However, now we assume that the Young's modulus T and the mass density ρ are constant and that our spatial interval is $[0, 1]$. Hence the model becomes

$$\frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(T\frac{\partial w}{\partial \zeta}(\zeta, t)\right), \qquad \zeta \in [0, 1], \tag{12.33}$$

$$u(t) = \begin{bmatrix} T\frac{\partial w}{\partial \zeta}(1, t) \\ T\frac{\partial w}{\partial \zeta}(0, t) \end{bmatrix}, \tag{12.34}$$

$$y(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(1, t) \\ \frac{\partial w}{\partial t}(0, t) \end{bmatrix}. \tag{12.35}$$

Here $w(\zeta, t)$ is the vertical position of the string at position ζ and time t. We write this system using the state variable $x = \begin{bmatrix} \rho\frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{bmatrix}$, and find

$$\frac{\partial x_1}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta}\left(Tx_2(\zeta, t)\right), \qquad \frac{\partial x_2}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta}\left(\frac{x_1(\zeta, t)}{\rho}\right), \tag{12.36}$$

$$u(t) = \begin{bmatrix} Tx_2(1, t) \\ Tx_2(0, t) \end{bmatrix}, \tag{12.37}$$

$$y(t) = \begin{bmatrix} \frac{1}{\rho}x_1(1, t) \\ \frac{1}{\rho}x_1(0, t) \end{bmatrix}. \tag{12.38}$$

In order to calculate the transfer function we have to solve the ordinary differential equation

$$sx_{10}(\zeta) = T\frac{dx_{20}}{d\zeta}(\zeta), \qquad sx_{20}(\zeta) = \frac{1}{\rho}\frac{dx_{10}}{d\zeta}(\zeta), \tag{12.39}$$

$$u_0 = \begin{bmatrix} u_{10} \\ u_{20} \end{bmatrix} = \begin{bmatrix} Tx_{20}(1) \\ Tx_{20}(0) \end{bmatrix}, \tag{12.40}$$

$$y_0 = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho}x_{10}(1) \\ \frac{1}{\rho}x_{10}(0) \end{bmatrix}. \tag{12.41}$$

It is easy to see that the solution of (12.39) is given by

$$x_{10}(\zeta) = \alpha e^{\lambda s \zeta} + \beta e^{-\lambda s \zeta}, \qquad x_{20}(\zeta) = \frac{\alpha\lambda}{\rho} e^{\lambda s \zeta} - \frac{\beta\lambda}{\rho} e^{-\lambda s \zeta}, \tag{12.42}$$

where $\lambda = \sqrt{\frac{\rho}{T}}$, and α, β are complex constants.

Using equation (12.40) we can relate the constants α and β to u_0,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\lambda}{e^{\lambda s} - e^{-\lambda s}} \begin{bmatrix} 1 & -e^{-\lambda s} \\ 1 & -e^{\lambda s} \end{bmatrix} u_0. \tag{12.43}$$

Combining this with (12.41), gives

$$y_0 = \frac{\lambda}{\rho(e^{\lambda s} - e^{-\lambda s})} \begin{bmatrix} e^{\lambda s} + e^{-\lambda s} & -2 \\ 2 & -e^{\lambda s} - e^{-\lambda s} \end{bmatrix} u_0.$$

Thus the transfer function is given by

$$G(s) = \frac{\lambda}{\rho} \begin{bmatrix} \frac{1}{\tanh(\lambda s)} & -\frac{1}{\sinh(\lambda s)} \\ \frac{1}{\sinh(\lambda s)} & -\frac{1}{\tanh(\lambda s)} \end{bmatrix}. \tag{12.44}$$

Using now the balance equation (11.38), we find

$$\begin{aligned} 2\mathrm{Re}\,(s)\|x_0\|^2 &= \mathrm{Re}\,(u_{10}^* y_{10}) - \mathrm{Re}\,(u_{20}^* y_{20}) \\ &= \mathrm{Re}\,(u_{10}^* G_{11}(s) u_{10} + u_{10}^* G_{12}(s) u_{20}) \\ &\quad - \mathrm{Re}\,(u_{20}^* G_{21}(s) u_{10} + u_{20}^* G_{22}(s) u_{20}). \end{aligned} \tag{12.45}$$

Choosing $u_{20} = 0$, we conclude that the real part of $G_{11}(s)$ is positive for $\mathrm{Re}(s) > 0$. An analytic function f on $\mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) > 0\}$ which satisfies $\mathrm{Re} f(s) \geq 0$ for $\mathrm{Re}(s) > 0$ and $f(s) \in \mathbb{R}$ for $s \in (0, \infty)$, is called *positive real.* Since physical parameters are real, we conclude from Theorem 12.2.1 that the transfer function is real-valued for real s. Using the fact that G_{11} is analytic on $s \in \mathbb{C}_0^+$, equation (12.45) implies that G_{11} is positive real. This can also be checked by direct calculation on $G_{11}(s) = \frac{\lambda}{\rho \tanh(\lambda s)}$.

Next we consider the system defined by the p.d.e. (12.33) with input $u(t) = T\frac{\partial w}{\partial \zeta}(0, t)$, output $y(t) = \frac{\partial w}{\partial t}(1, t)$ and boundary condition $T\frac{\partial w}{\partial \zeta}(1, t) = 0$. We could proceed as we did above. However, we can easily obtain the transfer function by choosing $u_{10} = 0$ in (12.40) and only considering y_{10} in (12.41). Hence the transfer function of this system is $-\frac{\lambda}{\rho \sinh(\lambda s)}$.

12.3 Exercises

12.1. Consider the vibrating string of Example 7.1.1. Thus in particular, we choose $\frac{\partial w}{\partial t}(a, t) = 0$. We control this system by applying a force at the right-hand side $u(t) = T(b)\frac{\partial w}{\partial \zeta}(b, t)$ and we measure the velocity at the same position, i.e., $y(t) = \frac{\partial w}{\partial t}(b, t)$.

(a) Show that the transfer function is positive real.
Hint: See Exercise 11.3

(b) Determine the transfer function for the case that the functions ρ and T are constant.

(c) Next choose $T(\zeta) = e^{\zeta}$, and $\rho(\zeta) = 1$, and determine the transfer function for this case.
Hint: You may use a computer package like Maple or Mathematica.

12.2. Consider the system of the transmission line of Exercise 7.1. As input we take the voltage at the right-hand side, and as output we take the current at the left-hand side. Furthermore, the voltage at the left-hand side is set to zero.

Determine the transfer function under the assumption that the capacity C and the inductance L are constant.

Hint: See also Exercise 11.2.

12.3. In this exercise we prove some well-known properties of transfer functions. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two systems, i.e, $\mathfrak{S}_1 \subset L^1_{\text{loc}}([0,\infty); U_1 \times R_1 \times Y_1)$ and $\mathfrak{S}_2 \subset L^1_{\text{loc}}([0,\infty); U_2 \times R_2 \times Y_2)$. Assume that for a given $s \in \mathbb{C}$ both systems have a transfer function. Furthermore, we assume that for both systems the output is determined by the input and the state, that is, if $(u_0 e^{st}, r_0 e^{st}, y(t))_{t\geq 0} \in \mathfrak{S}$, then $y(t) = y_0 e^{st}$ for some y_0 and for almost every $t \geq 0$.

(a) Show that for boundary control systems the output is determined by the input and the state, see Theorem 12.1.3.

(b) Assume that $Y_1 = U_2$. The *series connection* $\mathfrak{S}_{\text{series}} \subset L^1_{\text{loc}}([0,\infty); U_1 \times (R_1 \times R_2) \times Y_2)$ of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is defined as follows, see Figure 12.2,

$$(u_1, (r_1, r_2), y_2) \in \mathfrak{S}_{\text{series}} \text{ if there exists a } y_1 \text{ such that}$$
$$(u_1, r_1, y_1) \in \mathfrak{S}_1 \text{ and } (y_1, r_2, y_2) \in \mathfrak{S}_2.$$

Show that the series connection $\mathfrak{S}_{\text{series}}$ has the transfer function $G(s) = G_2(s)G_1(s)$ at s.

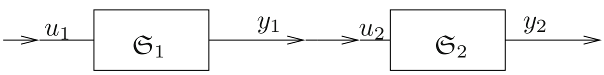

Figure 12.2: Series connection

(c) Assume that $U_1 = U_2$ and $Y_1 = Y_2$. The *parallel connection* $\mathfrak{S}_{\text{parallel}} \subset L^1_{\text{loc}}([0,\infty); U_1 \times (R_1 \times R_2) \times Y_2)$ of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is defined as follows:

$$(u_1, (r_1, r_2), y) \in \mathfrak{S}_{\text{parallel}} \text{ if there exists a } y_1 \in Y \text{ and } y_2 \in Y_2 \text{ such that}$$
$$(u_1, r_1, y_1) \in \mathfrak{S}_1, (u_1, r_2, y_2) \in \mathfrak{S}_2, \text{and } y = y_1 + y_2.$$

Show that the parallel connection $\mathfrak{S}_{\text{parallel}}$ has the transfer function $G_1(s) + G_2(s)$ at s.

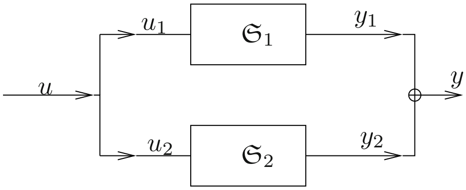

Figure 12.3: Parallel connection

(d) Assume that $U_1 = Y_2$ and $Y_1 = U_2$. The *feedback connection* $\mathfrak{S}_{\text{feedback}} \subset L^1_{\text{loc}}([0,\infty); U_1 \times (R_1 \times R_2) \times Y_1)$ of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is defined as follows:

$$(u,(r_1,r_2),y_1) \in \mathfrak{S}_{\text{feedback}} \text{ if there exists a } u_1 \text{ and } y_2 \text{ such that}$$
$$(u_1,r_1,y_1) \in \mathfrak{S}_1, (y_1,r_2,y_2) \in \mathfrak{S}_2, \text{and } u_1 = u - y_2.$$

Show that the feedback connection $\mathfrak{S}_{\text{feedback}}$ has the transfer function $G_1(s) \cdot (I + G_2(s)G_1(s))^{-1}$ at s, provided $I + G_2(s)G_1(s)$ is invertible.

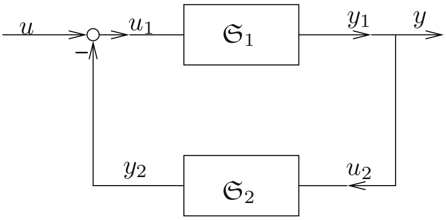

Figure 12.4: Feedback connection

12.4 Notes and references

The idea for defining the transfer function via exponentially solutions is old, but has hardly been investigated for distributed parameter systems. [64] was the first paper where this approach has been used for infinite-dimensional systems. We note that in this paper a transfer function is called a characteristic function. One may find the exponential solution in Polderman and Willems [47], where all solutions of this type are called the exponential behavior.

The formula for the transfer function, $G(s) = C(sI - A)^{-1}B + D$ can also easily be derived using the Laplace transform, see e.g. [10]. However, via the Laplace transform approach the function is only defined in some right half-plane and not on the whole resolvent set of A. In finite-dimensional spaces the transfer function is rational, and there are no mathematical difficulties. Therefore, the transfer function can easily be extended to $\rho(A)$. The situation is different for infinite-dimensional spaces, since transfer functions can contain terms like $\sqrt{s}$, and for these functions it is less clear how to extend them.

Positive realness is a well-studied property of transfer functions, see e.g. [1]. It first appeared in the study of electrical networks. Any (ideal) electrical network consisting of inductors, capacitors, and gyrators has a positive real transfer function, and even the converse holds, i.e., every (rational) positive real transfer function can be realized by such a network, see e.g. [2]. Our examples show that this property appears quite often and that positive realness is closely related to the energy balance.

Chapter 13

Well-posedness

The concept of well-posedness can easily be explained by means of the abstract linear system $\Sigma(A, B, C, D)$ introduced in Section 10.2, that is, for the set of equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{13.1}$$

$$y(t) = Cx(t) + Du(t), \tag{13.2}$$

where x is a X-valued function, u is an U-valued function and y is a Y-valued function. The spaces U, X and Y are assumed to be Hilbert spaces. Further, the operator A is assumed to be the infinitesimal generator of a C_0-semigroup, and B, C, and D are bounded linear operators. Under these assumptions the abstract differential equation (13.1)–(13.2) possesses for every $u \in L^1([0,\tau]; U)$ a unique (mild) solution, see Theorem 10.2.1. Since $L^2([0,\tau]; U) \subset L^1([0,\tau]; U)$ for $\tau < \infty$, the same assertion holds for $u \in L^2([0,\tau]; U)$. Existence of (mild) solutions for an arbitrary initial condition $x_0 \in X$ and an arbitrary input $u \in L^2([0,\tau]; U)$, such that x is continuous and $y \in L^2([0,\tau]; Y)$ is called *well-posedness*. Under the assumption that B, C, and D are bounded linear operators, the system (13.1)–(13.2) is well-posed if and only if A is the infinitesimal generator of a C_0-semigroup. The aim of this chapter is to extend this result to our class of port-Hamiltonian systems.

13.1 Well-posedness for boundary control systems

In this section we turn to the definition of well-posedness. Although well-posedness can be defined in a quite general setup, see the notes and references section, we restrict ourselves to the class of boundary control systems as introduced in Chapter 11. We showed that boundary control systems in general cannot be written in the form (13.1)–(13.2) with B, C, and D bounded. However, we know that for every initial condition and every smooth input function there exists a (unique)

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Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1_13,

mild solution of the state differential equation given by (11.9). Furthermore, from Theorem 11.2.1 follows that for smooth initial conditions and smooth inputs the output is well-defined. In the following example we see that the solution can often be extended to a larger set of initial conditions and input signals.

Example 13.1.1. Consider the controlled transport equation on the interval $[0, 1]$ with scalar control and observation on the boundary

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \quad x(\zeta, 0) = x_0(\zeta), \qquad \zeta \in [0, 1], \tag{13.3}$$

$$u(t) = x(1, t), \tag{13.4}$$

$$y(t) = x(0, t). \tag{13.5}$$

From Example 11.1.4 follows that the mild solution of (13.3)–(13.4) is given by

$$x(\zeta, t) = \begin{cases} x_0(\zeta + t), & \zeta + t \le 1, \\ u(\zeta + t - 1), & \zeta + t > 1. \end{cases} \tag{13.6}$$

For every $t \ge 0$ the function $x(\cdot, t)$ is an element of $X = L^2(0, 1)$, whenever $u \in L^2(0, \tau)$ and $x_0 \in X$. Furthermore, $x(\cdot, t)$ is a continuous function in t, i.e., $\|x(\cdot, t) - x(\cdot, t + h)\|$ converges to zero when h converges to zero, see Exercise 6.3. Hence the mild solution (13.6) can be extended from controls in $H^1(0, \tau)$ to $L^2(0, \tau)$. If x_0 and u are smooth, then we clearly see that $y(t)$ is well-defined for every $t \ge 0$ and it is given by

$$y(t) = \begin{cases} x_0(t), & 0 \le t \le 1, \\ u(t - 1), & t > 1. \end{cases} \tag{13.7}$$

However, if $x_0 \in L^2(0, 1)$ and $u \in L^2(0, \tau)$, then the expression (13.7) still implies that y is well-defined as an L^2-function.

Summarizing, we can define a (mild) solution for (13.3)–(13.5) for all $x_0 \in X$ and all $u \in L^2(0, \tau)$. This solution defines a continuous state trajectory in the state space, and an output trajectory which is square integrable on every compact time interval. Hence this system is well-posed.

In the previous example we showed well-posedness for a transport equation with control and observation at the boundary. Next we formally define well-posedness for the boundary control systems introduced in Section 11.1 and 11.2. Thus the system is given by

$$\dot{x}(t) = \mathfrak{A}x(t) \qquad x(0) = x_0, \tag{13.8}$$

$$\mathfrak{B}x(t) = u(t), \tag{13.9}$$

$$\mathfrak{C}x(t) = y(t). \tag{13.10}$$

As in Chapter 11 we need the following assumptions, see also Definition 11.1.1.

Assumption 13.1.2. The operators defining system (13.8)–(13.10) satisfy:

1. $\mathfrak{A} : D(\mathfrak{A}) \subset X \to X$, $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$, $\mathfrak{C} : D(\mathfrak{A}) \subset X \to Y$ are linear operators, $D(\mathfrak{A}) \subset D(\mathfrak{B})$, and X, U, Y are Hilbert spaces.
2. The operator $A : D(A) \to X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and
$$Ax = \mathfrak{A}x \qquad \text{for } x \in D(A) \tag{13.11}$$
is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X.
3. There exists an operator $B \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have $Bu \in D(\mathfrak{A})$, $\mathfrak{A}B \in \mathcal{L}(U, X)$ and
$$\mathfrak{B}Bu = u, \qquad u \in U. \tag{13.12}$$
4. The operator $\mathfrak{C}$ is bounded from the domain of A to Y. Here $D(A)$ is equipped with the graph norm.

We remark that Condition 4 was not required in Theorem 11.2.1. However, all the examples will satisfy this condition.

Definition 13.1.3. Consider the system (13.8)–(13.10) satisfying Assumption 13.1.2. We call this system *well-posed* if there exist a $\tau > 0$ and $m_\tau \geq 0$ such that for all $x_0 \in D(\mathfrak{A})$ and $u \in C^2([0, \tau]; U)$ with $u(0) = \mathfrak{B}x_0$ we have

$$\|x(\tau)\|_X^2 + \int_0^\tau \|y(t)\|^2 dt \leq m_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u(t)\|^2 dt \right). \tag{13.13}$$

In general it is not easy to show that a boundary control system is well-posed. However, there is a special class of systems for which well-posedness can be proved easily. This result is formulated next.

Proposition 13.1.4. *Assume that the boundary control system* (13.8)–(13.10) *satisfies Assumption* 13.1.2. *If every classical solution of the system satisfies*

$$\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2, \tag{13.14}$$

then the system is well-posed.

Proof. If we integrate equation (13.14) from 0 to τ, we find that

$$\|x(\tau)\|^2 - \|x(0)\|^2 = \int_0^\tau \|u(t)\|^2 \, dt - \int_0^\tau \|y(t)\|^2 \, dt,$$

which implies (13.13) with $m_\tau = 1$. □

In this special case the boundary control system satisfies (13.13) for every $\tau > 0$. In Theorem 13.1.7 we show that this holds for every well-posed boundary control system. Furthermore, if a boundary control system is well-posed, then we can define a (mild) solution of (13.8)–(13.10) for all $x_0 \in X$ and $u \in L^2([0,\tau];U)$ such that $t \mapsto x(t)$ is a continuous function in X and y is square integrable. To prove these statements we need the following lemmas for classical solutions of the system (13.8)–(13.10).

Lemma 13.1.5. *Assume that the boundary control system* (13.8)–(13.9) *satisfies conditions* 1, 2, *and* 3 *of Assumption* 13.1.2, $x_0 \in X$, *and* $u \in C^1([0,\tau];U)$. *Then the mild solution* x *of the boundary control system can written as*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\mathfrak{A}Bu(s)\,ds - A\int_0^t T(t-s)Bu(s)\,ds. \tag{13.15}$$

Proof. For $u \in C^1([0,\tau];U)$ Corollary 10.1.4 implies

$$T(t)Bu(0) + \int_0^t T(t-s)B\dot{u}(s)\,ds - Bu(t) = A\int_0^t T(t-s)Bu(s)\,ds.$$

Combining this with the formula for the mild solution, see (11.9), we obtain (13.15). □

Formula (13.15) holds in particular for classical solutions of boundary control systems if $x_0 \in D(\mathfrak{A})$, and $u \in C^2([0,t_1];U)$ with $\mathfrak{B}x_0 = u(0)$. For the proof of Theorem 13.1.7 we need certain properties of these classical solutions. The proof is left as an exercise to the reader, see Exercise 13.5.

Lemma 13.1.6. *Consider the boundary control system* (13.8)–(13.10) *satisfying conditions* 1, 2, *and* 3 *of Assumption* 13.1.2, *and let* $t_1 > 0$. *For* $x_0 \in D(\mathfrak{A})$, *and* $u \in C^2([0,t_1];U)$ *with* $\mathfrak{B}x_0 = u(0)$ *we denote the corresponding classical solution by* $x(\cdot;x_0,u)$ *and* $y(t;x_0,u)$, *see Theorem* 11.2.1, *respectively. Then the following results hold:*

1. *The functions* $x(\cdot;x_0,u)$ *and* $y(\cdot;x_0,u)$ *depend linearly on the initial condition and on the input, i.e., for all* $x_1, x_2 \in D(\mathfrak{A})$ *and* $u_1, u_2 \in C^2([0,t_1];U)$ *with* $\mathfrak{B}x_1 = u_1(0), \mathfrak{B}x_2 = u_2(0)$ *and* $\alpha, \beta \in \mathbb{K}$ *there holds*

$$x(t;\alpha x_1 + \beta x_2, \alpha u_1 + \beta u_2) = \alpha x(t;x_1,u_1) + \beta x(t;x_2,u_2), \qquad t \in [0,t_1] \tag{13.16}$$

   and a similar property holds for $y(t;x_0,u)$.

2. *The system is* causal, *i.e., if* $x_0 \in D(\mathfrak{A})$, *and* $u_1 \in C^2([0,t_1];U)$ *with* $\mathfrak{B}x_0 = u_1(0)$ *and if* $u_2 \in C^2([0,t_1];U)$ *satisfies* $u_1(t) = u_2(t)$ *for* $0 \le t \le \tau < t_1$, *then*

$$x(t;x_0,u_1) = x(t;x_0,u_2), \qquad t \in [0,\tau], \tag{13.17}$$

$$y(t;x_0,u_1) = y(t;x_0,u_2), \qquad t \in [0,\tau]. \tag{13.18}$$

3. *For* $x_0 \in D(\mathfrak{A})$, $u \in C^2([0,t_1];U)$ *with* $\mathfrak{B}x_0 = u(0)$ *there holds*

$$x(\tau_1+\tau_2; x_0, u) = x(\tau_1; x(\tau_2), u(\cdot+\tau_2)), \tag{13.19}$$
$$y(\tau_1+\tau_2; x_0, u) = y(\tau_1; x(\tau_2), u(\cdot+\tau_2)) \tag{13.20}$$

where

$$x(\tau_2) = x(\tau_2; x_0, u), \tag{13.21}$$

with $\tau_1, \tau_2 > 0$, *and* $\tau_1+\tau_2 \le t_1$.

The following theorem shows that if (13.13) holds for some $\tau > 0$, then (13.13) holds for all $\tau > 0$. However, note that the constant m_τ depends on τ.

Theorem 13.1.7. *If the boundary control system* (13.8)–(13.10) *satisfying Assumption* 13.1.2 *is well-posed, then for all* $\tau > 0$ *there exists a constant* $m_\tau > 0$ *such that* (13.13) *holds.*

Proof. Step 1. Using Theorem 11.2.1 and Lemma 13.1.6, for every $t > 0$ we can define the linear operator $\mathcal{S}(t) : D(\mathcal{S}(t)) \subset X \oplus L^2([0,t];U) \to X \oplus L^2([0,t];Y)$ by

$$\mathcal{S}(t)\begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} x(t;x_0,u) \\ y(t;x_0,u) \end{bmatrix}, \tag{13.22}$$

with

$$D(\mathcal{S}(t)) = \left\{ \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in X \oplus L^2([0,t];U) \mid x_0 \in D(\mathfrak{A}), u \in C^2([0,t];U), \mathfrak{B}x_0 = u(0) \right\}. \tag{13.23}$$

Here x is the classical solution of the boundary control system and y is the corresponding output. For $\left[\begin{smallmatrix} x \\ f \end{smallmatrix}\right] \in X \oplus L^2([0,t];Z)$ with Z a Hilbert space, we define the norm as

$$\left\| \begin{bmatrix} x \\ f \end{bmatrix} \right\|^2_{X,L^2} = \|x\|_X^2 + \int_0^t \|f(t)\|_Z^2 dt.$$

As the boundary control system is well-posed there exist $\tau_0 > 0$ and $m_{\tau_0} \ge 0$ such that

$$\left\| \mathcal{S}(\tau_0) \begin{bmatrix} x_0 \\ u \end{bmatrix} \right\|^2_{X,L^2} = \left\| \begin{bmatrix} x(\tau_0;x_0,u) \\ y(\tau_0;x_0,u) \end{bmatrix} \right\|^2_{X,L^2} \le m_{\tau_0} \left\| \begin{bmatrix} x_0 \\ u \end{bmatrix} \right\|^2_{X,L^2}$$

for $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in D(\mathcal{S}(\tau_0))$. Since $\mathcal{S}(\tau_0)$ is densely defined, there exists a unique bounded extension $\overline{\mathcal{S}(\tau_0)}$ with norm less than or equal to $\sqrt{m_{\tau_0}}$. Equation (13.22) implies that the extension $\overline{\mathcal{S}(\tau_0)}$ can be written as

$$\overline{\mathcal{S}(\tau_0)} = \begin{bmatrix} \overline{\mathcal{S}_1(\tau_0)} \\ \overline{\mathcal{S}_2(\tau_0)} \end{bmatrix}, \text{ with } \mathcal{S}(\tau_0) = \begin{bmatrix} \mathcal{S}_1(\tau_0) \\ \mathcal{S}_2(\tau_0) \end{bmatrix}. \tag{13.24}$$

Step 2. In this step we prove that for $\lambda \in \rho(A)$ and $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in X \oplus L^2([0,t];U)$ the following holds:

$$(\lambda I - A)^{-1}\overline{\mathcal{S}_1(\tau_0)}\begin{bmatrix} x_0 \\ u \end{bmatrix} = T(\tau)(\lambda I - A)^{-1}x_0 \\ + \int_0^\tau T(\tau - s)\left((\lambda I - A)^{-1}\mathfrak{A}B - A(\lambda I - A)^{-1}B\right)u(s)\,ds. \tag{13.25}$$

Using the definition of $\mathcal{S}_1(\tau_0)$, see (13.24), and Lemma 13.1.5, (13.25) holds for $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in D(\mathcal{S}(\tau_0))$. Since $D(\mathcal{S}(\tau_0))$ is dense in $X \oplus L^2([0,\tau_0];U)$, and since both sides of equation (13.25) are well-defined for $x_0 \in X$ and $u \in L^2([0,\tau_0];U)$ we conclude that this equality holds for all initial conditions and all input functions.

For functions f, $g \in L^2$, we define

$$(f \underset{\tau}{\diamondsuit} g)(t) = \begin{cases} f(t), & t < \tau, \\ g(t-\tau), & t > \tau. \end{cases} \tag{13.26}$$

Step 3. Next we show that $\mathcal{S}_1(\tau)$ has a bounded extension for $\tau \in (0,\tau_0)$. Let $\tau \in (0,\tau_0)$ be arbitrary. We define the operator $\overline{\mathcal{S}_1(\tau)}$ by

$$\overline{\mathcal{S}_1(\tau)}\begin{bmatrix} x_0 \\ u \end{bmatrix} := T(\tau)x_0 + \overline{\mathcal{S}_1(\tau_0)}\begin{bmatrix} 0 \\ 0 \underset{\tau_0-\tau}{\diamondsuit} u \end{bmatrix}. \tag{13.27}$$

Note that $\overline{\mathcal{S}_1(\tau)}$ is defined by (13.27) and that we will only show in the sequel that $\overline{\mathcal{S}_1(\tau)}$ is indeed the bounded extension of $\mathcal{S}_1(\tau)$. Clearly $\overline{\mathcal{S}_1(\tau)}$ is a bounded operator from $X \oplus L^2([0,\tau];U)$ to X. It remains to show that $\overline{\mathcal{S}_1(\tau)}$ is an extension of $\mathcal{S}_1(\tau)$.

For $\lambda \in \rho(A)$ and $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in X \oplus L^2([0,\tau];U)$, (13.27) and (13.25) imply

$$\begin{aligned} &(\lambda I - A)^{-1}\overline{\mathcal{S}_1(\tau)}\begin{bmatrix} x_0 \\ u \end{bmatrix} \\ &= (\lambda I - A)^{-1}T(\tau)x_0 + (\lambda I - A)^{-1}\overline{\mathcal{S}_1(\tau_0)}\begin{bmatrix} 0 \\ 0 \underset{\tau_0-\tau}{\diamondsuit} u \end{bmatrix} \\ &= (\lambda I - A)^{-1}T(\tau)x_0 \\ &\quad + \int_0^{\tau_0} T(\tau_0 - s)\left((\lambda I - A)^{-1}\mathfrak{A}B - A(\lambda I - A)^{-1}B\right)(0 \underset{\tau_0-\tau}{\diamondsuit} u)(s)\,ds \\ &= (\lambda I - A)^{-1}T(\tau)x_0 \\ &\quad + \int_0^{\tau} T(\tau - s)\left((\lambda I - A)^{-1}\mathfrak{A}B - A(\lambda I - A)^{-1}B\right)u(s)\,ds. \end{aligned} \tag{13.28}$$

Using Corollary 10.1.4, for $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in D(\mathcal{S}(\tau))$ the integral maps into $D(A)$, and thus we obtain

$$(\lambda I - A)^{-1}\overline{\mathcal{S}_1(\tau)}\begin{bmatrix} x_0 \\ u \end{bmatrix} = (\lambda I - A)^{-1}T(\tau)x_0$$
$$+ (\lambda I - A)^{-1}\int_0^\tau T(\tau - s)\mathfrak{A}Bu(s)\,ds - (\lambda I - A)^{-1}A\int_0^\tau T(\tau - s)Bu(s)\,ds.$$

Multiplying from the left by $(\lambda I - A)$, we obtain that, for $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in D(\mathcal{S}(\tau))$,

$$\overline{\mathcal{S}_1(\tau)}\begin{bmatrix} x_0 \\ u \end{bmatrix} = T(\tau)x_0 + \int_0^\tau T(\tau - s)\mathfrak{A}Bu(s)\,ds - A\int_0^\tau T(\tau - s)Bu(s)\,ds. \quad (13.29)$$

By Lemma 13.1.5, the right-hand side equals the classical state trajectory of the system (13.8)–(13.10). Thus by the definition of $\mathcal{S}(\tau)$, see (13.22), we obtain

$$\overline{\mathcal{S}_1(\tau)}\begin{bmatrix} x_0 \\ u \end{bmatrix} = \mathcal{S}_1(\tau)\begin{bmatrix} x_0 \\ u \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ u \end{bmatrix} \in D(\mathcal{S}(\tau)).$$

Thus $\overline{\mathcal{S}_1(\tau)}$ is the unique bounded extension of $\mathcal{S}_1(\tau)$ to $X \oplus L^2([0,\tau];U)$.

Step 4. Let $\tau \in (0, \tau_0)$. We assume that $\left[\begin{smallmatrix} x_0 \\ u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_0 \\ u_2 \end{smallmatrix}\right] \in D(\mathcal{S}(\tau_0))$ satisfy $u_1(t) = u_2(t)$ for all $t \in [0,\tau]$. By Lemma 13.1.6, we obtain

$$\mathcal{S}_2(\tau_0)\begin{bmatrix} x_0 \\ u_1 \end{bmatrix}(t) = \mathcal{S}_2(\tau_0)\begin{bmatrix} x_0 \\ u_2 \end{bmatrix}(t), \qquad t \in [0,\tau]. \quad (13.30)$$

Thus the bounded extension $\overline{\mathcal{S}_2(\tau_0)}$ satisfies for all $x_0 \in X$ and $u_1, u_2 \in L^2([0,\tau_0];U)$ with $u_1(t) = u_2(t)$ a.e. for $t \in [0,\tau]$,

$$\overline{\mathcal{S}_2(\tau_0)}\begin{bmatrix} x_0 \\ u_1 \end{bmatrix}(t) = \overline{\mathcal{S}_2(\tau_0)}\begin{bmatrix} x_0 \\ u_2 \end{bmatrix}(t), \qquad \text{for a.e. } t \in [0,\tau]. \quad (13.31)$$

Step 5. Let $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in D(\mathcal{S}(\tau))$ and define $u_{\text{ext}} \in C^2([0,\tau_0];U)$ as a C^2-extension of u, i.e., $u_{\text{ext}}(t) = u(t)$ for $t \in [0,\tau]$. Then it is easy to see that $\left[\begin{smallmatrix} x_0 \\ u_{\text{ext}} \end{smallmatrix}\right] \in D(\mathcal{S}(\tau_0))$. By Theorem 11.2.1 for $t \in [0,\tau]$,

$$\left(\mathcal{S}_2(\tau)\begin{bmatrix} x_0 \\ u \end{bmatrix}\right)(t) = \left(\mathcal{S}_2(\tau_0)\begin{bmatrix} x_0 \\ u_{\text{ext}} \end{bmatrix}\right)(t). \quad (13.32)$$

Using (13.31), (13.32), and the estimate (13.13) we obtain

$$\begin{aligned}
\left\|\mathcal{S}_2(\tau)\begin{bmatrix} x_0 \\ u \end{bmatrix}\right\|^2_{L^2(0,\tau)} = \left\|\mathcal{S}_2(\tau_0)\begin{bmatrix} x_0 \\ u_{\text{ext}} \end{bmatrix}\right\|^2_{L^2(0,\tau)} &= \left\|\overline{\mathcal{S}_2(\tau_0)}\begin{bmatrix} x_0 \\ u \mathbin{\underset{\tau}{\Diamond}} 0 \end{bmatrix}\right\|^2_{L^2(0,\tau)} \\
&\le \left\|\overline{\mathcal{S}_2(\tau_0)}\begin{bmatrix} x_0 \\ u \mathbin{\underset{\tau}{\Diamond}} 0 \end{bmatrix}\right\|^2_{L^2(0,\tau_0)} \\
&\le m_{\tau_0}\left(\|x_0\|^2 + \|u \mathbin{\underset{\tau}{\Diamond}} 0\|^2_{L^2(0,\tau_0)}\right) \\
&= m_{\tau_0}\left(\|x_0\|^2 + \|u\|^2_{L^2(0,\tau)}\right).
\end{aligned}$$

Thus $\mathcal{S}_2(\tau)$ possesses a linear bounded extension from $X \oplus L^2([0,\tau];U)$ to $L^2([0,\tau];Y)$. Combining this fact with the result of Step 3, we may conclude that (13.13) holds for τ with $m_\tau = \|\mathcal{S}(\tau)\|$.

Step 6. Next we show that (13.13) holds for $\tau > \tau_0$. We first assume that $\tau \in (\tau_0, 2\tau_0]$, and we write $\tau = \tau_0 + t_1$ with $t_1 \in (0,\tau_0]$. For $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in D(\mathcal{S}(\tau))$ it is easy to see that $\left[\begin{smallmatrix} x_0 \\ u|_{[0,\tau_0]} \end{smallmatrix}\right] \in D(\mathcal{S}(\tau_0))$. Using equations (13.19)–(13.21), the following equations hold:

$$\mathcal{S}_1(\tau) \left[\begin{array}{c} x_0 \\ u \end{array} \right] = \mathcal{S}_1(t_1) \left[\begin{array}{c} x(\tau_0) \\ u(\tau_0 + \cdot) \end{array} \right], \tag{13.33}$$

$$\left(\mathcal{S}_2(\tau) \left[\begin{array}{c} x_0 \\ u \end{array} \right] \right)(t) = \begin{cases} \left(\mathcal{S}_2(\tau_0) \left[\begin{array}{c} x_0 \\ u \end{array} \right] \right)(t), & t \le \tau_0, \\ \left(\mathcal{S}_2(t_1) \left[\begin{array}{c} x(\tau_0) \\ u(\tau_0 + \cdot) \end{array} \right] \right)(t), & t \in (\tau_0, \tau], \end{cases} \tag{13.34}$$

where

$$x(\tau_0) = S_1(\tau_0) \left[\begin{array}{c} x_0 \\ u|_{[0,\tau_0]} \end{array} \right]. \tag{13.35}$$

Using these equations and the boundedness of the operators $\overline{\mathcal{S}_1(t)}, \overline{\mathcal{S}_2(t)}$, $t \in (0,\tau_0]$, we may conclude that the operators $\mathcal{S}_1(\tau)$ and $\mathcal{S}_2(\tau)$ possess bounded extensions from $X \oplus L^2([0,\tau];U)$ to X and to $L^2([0,\tau];Y)$, respectively. Therefore, (13.13) holds for τ with $m_\tau = \|\mathcal{S}(\tau)\|$. The general case $\tau > 2\tau_0$ follows by induction. □

In the proof of the previous theorem we have introduced the operators $\mathcal{S}(t)$ mapping the initial condition and input to the state at time t and the output on the time interval $[0,t]$. If the boundary control system is well-posed, then we may extend $\mathcal{S}(t)$ to a bounded linear operator from $X \oplus L^2([0,t];U)$ to $X \oplus L^2([0,t];Y)$ and we can decompose the operator $\overline{\mathcal{S}(t)}$ as

$$\overline{\mathcal{S}(t)} = \left[\begin{array}{cc} \mathcal{S}_{11}(t) & \mathcal{S}_{12}(t) \\ \mathcal{S}_{21}(t) & \mathcal{S}_{22}(t) \end{array} \right]. \tag{13.36}$$

Equation (13.27) implies that $\mathcal{S}_{11}(t)$ equals the semigroup at time t. We are now in the position to extend the definition of a mild solution, see Section 11.1.

Definition 13.1.8. Let (13.8)–(13.10) be a well-posed boundary control system with input space U and output space Y and let $\overline{\mathcal{S}(t)}$ be decomposed as in (13.36). For $x \in X$ and $u \in L^2([0,\tau];U)$ the *mild solution of* (13.8)–(13.10) is given by

$$\begin{aligned} x(t) &= \mathcal{S}_{11}(t)x_0 + \mathcal{S}_{12}(t)u, \qquad t \in [0,\tau], \\ y &= \mathcal{S}_{21}(\tau)x_0 + \mathcal{S}_{22}(\tau)u. \end{aligned} \tag{13.37}$$

Remark 13.1.9. For a fixed $\tau > 0$ it follows from the proof of Theorem 13.1.7 that there exists a constant $m_\tau \geq 0$ such that for every $t \in (0, \tau]$ we have $\|\mathcal{S}_{12}(t)u\| \leq m_\tau \|u\|$. Combining this observation with the fact that the classical solutions are continuous, it is easy to show that the state trajectory x of a mild solution of (13.8)–(13.10) is continuous. Note that the output is only square integrable. The mild state space solution defined in Definition 13.1.8 extends the mild solution as defined by (11.9). The fact that $\mathcal{S}_{11}(t)x_0 + \mathcal{S}_{12}(t)u$ equals (11.9) for $x_0 \in X$ and $u \in H^1([0, \tau]; U)$ can be shown as in Step 3 of the proof of Theorem 13.1.7.

In Chapter 12 we showed that the boundary control system (13.8)–(13.10) possesses a transfer function. Using Theorem 12.1.3 we conclude that this function is defined on the resolvent set of A. Since A generates a C_0-semigroup, the resolvent set contains a right half-plane, see Proposition 5.2.4, which implies that the transfer function exists on some right-half plane. Furthermore, the transfer function is bounded as $\mathrm{Re}(s) \to \infty$ for a well-posed system.

Lemma 13.1.10. *If G is the transfer function of a well-posed system, then*

$$\|G(s)\| \leq \sqrt{m_\tau} \tag{13.38}$$

for $Re(s) \geq \frac{\log(m_\tau)}{2\tau}$, where τ and m_τ are the constants of equation (13.13).

Proof. Choose $s \in \mathbb{C}$ such that $m_\tau \leq |e^{2s\tau}|$, i.e., $\mathrm{Re}(s) \geq \frac{\log(m_\tau)}{2\tau}$. For this fixed $s \in \mathbb{C}$ and an arbitrary $u_0 \in U$ we consider the exponential solution $(u_0 e^{st}, x_0 e^{st}, G(s)u_0 e^{st})_{t \geq 0}$. Inequality (13.13) implies that

$$\begin{aligned} |e^{2s\tau}| \|x_0\|_X^2 + \int_0^\tau \|G(s)u_0\|^2 |e^{st}|^2 dt &\leq m_\tau \left(\|x_0\|_X^2 + \int_0^\tau \|u_0\|^2 |e^{st}|^2 dt \right) \\ &\leq |e^{2s\tau}| \|x_0\|_X^2 + m_\tau \int_0^\tau \|u_0\|^2 |e^{st}|^2 dt. \end{aligned}$$

This inequality shows that $\|G(s)u_0\|^2 \leq m_\tau \|u_0\|^2$. Since u_0 is arbitrary, we conclude that (13.38) holds. □

Although the transfer function is bounded on some right half-plane, this does not imply that the transfer function converges along the real axis. If it does converge, then the boundary control system is called regular.

Definition 13.1.11. Let G be the transfer function of a well-posed boundary control system. The boundary control system is called *regular* if $\lim_{s \in \mathbb{R}, s \to \infty} G(s)$ exists. If the boundary control system is regular, then the *feed-through* term D is defined as $D = \lim_{s \in \mathbb{R}, s \to \infty} G(s)$.

Well-posed regular boundary control systems are closed under a class of output feedback operators and bounded perturbations of the generator. These results were first obtained by G. Weiss. The proofs of these result lie outside the scope of this chapter, and we refer the interested reader to [60] or [51].

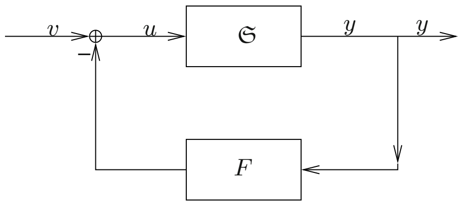

Figure 13.1: The closed loop system

Theorem 13.1.12. *Assume that the boundary system* (13.8)–(13.10) *satisfying Assumption* 13.1.2*, denoted by* $\mathfrak{S}$*, is well-posed. We denote the corresponding transfer function by* G*. Let* F *be a bounded linear operator from* Y *to* U *and assume that the inverse of* $I+G(s)F$ *exists and is bounded for* s *in some right half-plane. Then the closed loop system as depicted in Figure* 13.1 *is again well-posed, that is, the boundary control system*

$$\dot{x}(t) = \mathfrak{A}x(t) \tag{13.39}$$

$$v(t) = (\mathfrak{B} + F\mathfrak{C})x(t) \tag{13.40}$$

$$y(t) = \mathfrak{C}x(t) \tag{13.41}$$

satisfies Assumption 13.1.2 *and it is well-posed. If the boundary control system* $\mathfrak{S}$ *is regular, then the closed loop system is regular as well. The operator* F *is called a* feedback operator*.*

Remark 13.1.13. A feedback operator can be used to stabilize a system. More precisely, a feedback operator changes the operator $\mathfrak{B}$, that is, if we start with an unstable system, then via a clever choice of the feedback operator F we may stabilize the system. For instance, in Example 9.2.1 the boundary condition (9.28) can be interpreted as a feedback relation. Namely, the input (force at the right-hand side) is equal to the feedback, $-k$, times the output (velocity at the right-hand side).

Lemma 13.1.14. *Consider the boundary system* (13.8)–(13.10) *satisfying the conditions of Assumption* 13.1.2*, and let* Q *be a bounded operator on* X*, i.e.,* $Q \in \mathcal{L}(X)$*. This system is well-posed if and only if the system*

$$\dot{x}(t) = \mathfrak{A}x(t) + Qx(t) \tag{13.42}$$

with inputs and outputs given by (13.9)–(13.10) *is well-posed.*

Let G *denote the transfer function of* (13.8)–(13.10) *and* G_Q *the transfer function of* (13.42) *with* (13.9)–(13.10)*. Then*

$$\lim_{s\in\mathbb{R}, s\to\infty} G(s) = \lim_{s\in\mathbb{R}, s\to\infty} G_Q(s), \tag{13.43}$$

and

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = \lim_{\mathrm{Re}(s)\to\infty} G_Q(s). \tag{13.44}$$

13.2 Well-posedness for port-Hamiltonian systems

We now turn our attention to the class of port-Hamiltonian systems with boundary control and boundary observation as introduced in Sections 11.3, that is, to systems of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0(\mathcal{H}(\zeta)x(\zeta,t)), \tag{13.45}$$

$$u(t) = W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}, \tag{13.46}$$

$$0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}, \tag{13.47}$$

$$y(t) = W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}. \tag{13.48}$$

We assume that P_1, $\mathcal{H}$ and $W_B := \left[\begin{smallmatrix} W_{B,1} \\ W_{B,2}\end{smallmatrix}\right]$ satisfy Assumption 11.3.1. Thus in particular, for a.e. $\zeta \in [a,b]$, $\mathcal{H}(\zeta)$ is a self-adjoint $n\times n$ matrix satisfying $0 < mI \le \mathcal{H}(\zeta) \le MI$. Furthermore, $W_B := \left[\begin{smallmatrix} W_{B,1} \\ W_{B,2}\end{smallmatrix}\right]$ is a full rank matrix of size $n \times 2n$. We assume that $W_{B,1}$ is a $m \times 2n$ matrix[1]. Recall that the state space is given by the Hilbert space $X = L^2([a,b];\mathbb{K}^n)$ with the inner product

$$\langle f,g\rangle_X = \frac{1}{2}\int_a^b f(\zeta)^*\mathcal{H}(\zeta)g(\zeta)d\zeta. \tag{13.49}$$

The following lemma follows easily from Theorems 11.3.2 and 11.3.5, see also Exercise 11.4.

Lemma 13.2.1. *Let τ be a positive real number. Assume that the operator $A := P_1\frac{\partial}{\partial\zeta}\mathcal{H} + P_0\mathcal{H}$ with domain*

$$D(A) = \left\{x_0 \in X \mid \mathcal{H}x_0 \in H^1([a,b];\mathbb{K}^n), W_B\left[\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0}\end{smallmatrix}\right] = 0\right\}$$

is the infinitesimal generator of a C_0-semigroup on X. Then the system (13.45)–(13.47) *is a boundary control system satisfying Assumption* 13.1.2.

In particular, for every $\mathcal{H}x_0 \in H^1([a,b];\mathbb{K}^n)$ and every $u \in C^2([0,\tau];\mathbb{K}^m)$ with $u(0) = W_{B,1}\left[\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0}\end{smallmatrix}\right]$, and $0 = W_{B,2}\left[\begin{smallmatrix} f_{\partial,0} \\ e_{\partial,0}\end{smallmatrix}\right]$, there exists a unique classical solution of (13.45)–(13.47) *on* $[0,\tau]$. *Furthermore, the output* (13.48) *is well-defined and y is continuous on* $[0,\tau]$.

By Lemma 13.2.1, if A generates a C_0-semigroup on X, then the system possesses classical solutions for smooth inputs and initial conditions. Well-posedness implies that there exist solutions for every initial condition and every square integrable input. Now we are in the position to formulate our main result.

[1]Note that m has two meanings: It is used as a lower-bound for $\mathcal{H}$ and as the dimension of our input space.

Theorem 13.2.2. *Consider the port-Hamiltonian system* (13.45)–(13.48) *and assume that the conditions of Assumption* 11.3.1 *are satisfied. Furthermore, we assume that*

1. *The multiplication operator $P_1\mathcal{H}$ can be written as*

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta), \qquad \zeta \in [a,b], \tag{13.50}$$

   where Δ is a diagonal matrix-valued function, S is a matrix-valued function and both Δ and S are continuously differentiable on $[a,b]$.

2. $\operatorname{rank}\begin{bmatrix} W_{B,1} \\ W_{B,2} \\ W_C \end{bmatrix} = n + \operatorname{rank}(W_C)$.

Let X be the Hilbert space $L^2([a,b];\mathbb{K}^n)$ with inner product (13.49)*. If the operator A corresponding to the homogeneous p.d.e., i.e., $u \equiv 0$, generates a C_0-semigroup on X, then the system* (13.45)–(13.48) *is regular, and thus in particular well-posed. Furthermore, we have that $\lim_{\mathrm{Re}(s)\to\infty} G(s) = \lim_{s\to\infty, s\in\mathbb{R}} G(s)$.*

The proof of Theorem 13.2.2 will be given in Sections 13.3 and 13.4.

Assumption 11.3.1 and the Assumptions 1 and 2 of Theorem 13.2.2 can be easily checked. Thus it basically remains to check that A generates a C_0-semigroup. Note that this operator equals the operator A defined in Lemma 13.2.1. From Chapter 7 we know that if $W_B \Sigma W_B^* \geq 0$ and $P_0 = -P_0^*$, then A generates a (contraction) semigroup, and thus in this situation the system is well-posed. In particular, we obtain a mild solution for all square integrable inputs.

Remark 13.2.3.

1. Assumption 11.3.1 is very standard, and is assumed to be satisfied for all our port-Hamiltonian systems until now.

2. Note that we do not have any assumption on P_0. In fact the term $P_0\mathcal{H}$ may be replaced by any bounded operator on X. This result will be proved in Section 13.4.

3. Assumption 1 of Theorem 13.2.2 concerning the multiplication operator $P_1\mathcal{H}$ is not very strong, and will almost always be satisfied if $\mathcal{H}$ is continuously differentiable. Note that Δ contains the eigenvalues of $P_1\mathcal{H}$, whereas S^{-1} contains the eigenvectors.

4. The last condition implies that we are not measuring quantities that are set to zero, or chosen to be an input. This condition is not important for the proof, and will normally follow from correct modeling.

As mentioned above, Assumption 1 of Theorem 13.2.2 is quite weak. We illustrate this by means of an example, the wave equation studied in Example 7.1.1.

Example 13.2.4. From Example 7.1.1 together with equation (7.5) follows that the port-Hamiltonian formulation of the wave equation is given by

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} \right), \tag{13.51}$$

where $x_1 = \rho \frac{\partial w}{\partial t}$ is the momentum and $x_2 = \frac{\partial w}{\partial \zeta}$ is the strain.

Hence we have that

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}.$$

Being physical constants, the Young's modulus T and the mass density ρ are positive. Hence P_1 and $\mathcal{H}$ satisfy the first two conditions of Assumption 11.3.1. Under the assumption that T and ρ are continuously differentiable, we show that (13.50) holds. The eigenvalues of

$$P_1\mathcal{H}(\zeta) = \begin{bmatrix} 0 & T(\zeta) \\ \frac{1}{\rho(\zeta)} & 0 \end{bmatrix} \tag{13.52}$$

are given by $\pm\gamma(\zeta)$ with $\gamma(\zeta) = \sqrt{\frac{T(\zeta)}{\rho(\zeta)}}$. The corresponding eigenvectors are given by

$$\begin{bmatrix} \gamma(\zeta) \\ \frac{1}{\rho(\zeta)} \end{bmatrix} \text{ and } \begin{bmatrix} -\gamma(\zeta) \\ \frac{1}{\rho(\zeta)} \end{bmatrix}. \tag{13.53}$$

Hence

$$P_1\mathcal{H} = S^{-1}\Delta S = \begin{bmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{bmatrix}, \tag{13.54}$$

where we have omitted the dependence on ζ. This shows that the assumptions of Theorem 13.2.2 are satisfied.

In Example 13.2.4 the matrix $P_1\mathcal{H}$ possesses only real eigenvalues and the number of positive and negative eigenvalues is independent of ζ. This can be shown to hold in general.

Lemma 13.2.5. *Let P_1 and $\mathcal{H}$ satisfy the conditions of Theorem* 13.2.2*. Then Δ can be written as*

$$\Delta = \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix}, \tag{13.55}$$

where Λ is a diagonal real matrix-valued function, with (strictly) positive functions on the diagonal, and Θ is a diagonal real matrix-valued function, with (strictly) negative functions on the diagonal.

Proof. Since $\mathcal{H}(\zeta) > mI$ for every $\zeta \in [a, b]$, the square root of $\mathcal{H}(\zeta)$ exists for every $\zeta \in [a, b]$. By Sylvester's law of inertia, the inertia of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ equals the inertia of P_1. This implies that the inertia of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ is independent of ζ. Thus, since P_1 is invertible, zero is not an eigenvalue of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ and the number of negative eigenvalues of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ equals the number of negative eigenvalues of P_1. A similar statement holds for the positive eigenvalues.

A simple calculation shows that the eigenvalues of $\mathcal{H}(\zeta)^{\frac{1}{2}} P_1 \mathcal{H}(\zeta)^{\frac{1}{2}}$ are equal to the eigenvalues of $P_1\mathcal{H}(\zeta)$. Concluding, for every $\zeta \in [a, b]$ zero is not an eigenvalue of $P_1\mathcal{H}(\zeta)$, and the number of negative and positive eigenvalues of $P_1\mathcal{H}(\zeta)$ is independent of ζ. By regrouping the eigenvalues, we obtain the requested representation (13.55). □

The proof of Theorem 13.2.2 will be given in Sections 13.3 and 13.4. As can be guessed from the condition (13.50) we will diagonalize $P_1\mathcal{H}$. Therefore it is essential to know the systems properties for the case that $P_1\mathcal{H}$ is diagonal. In this situation it suffices to understand the systems behavior for a positive and a negative element. This is the subject of the following two lemmas.

Lemma 13.2.6. *Let λ be a positive continuous function on the interval $[a, b]$. Then the port-Hamiltonian system*

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta}\left(\lambda(\zeta) w(\zeta, t)\right), \qquad w(\zeta, 0) = w_0(\zeta), \qquad \zeta \in [a, b], \tag{13.56}$$

$$u(t) = \lambda(b) w(b, t), \tag{13.57}$$

$$y(t) = \lambda(a) w(a, t), \tag{13.58}$$

is a regular system on the state space $L^2(a, b)$ and the corresponding transfer function G satisfies

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = 0. \tag{13.59}$$

Proof. It is easy to see that the system (13.56)–(13.58) is a very simple version of the general Port-Hamiltonian system (13.45)–(13.48) with $P_1 = 1$, $\mathcal{H} = \lambda$, $W_B = [\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}]$ and $W_C = [-\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}]$. Since $W_B \Sigma W_B^* = 1 > 0$, we conclude that the operator A corresponding to the homogeneous p.d.e., i.e., $u \equiv 0$, generates a C_0-semigroup on $L^2(a, b)$. Thus Theorem 11.3.5 implies that (13.56)–(13.58) has a well-defined solution provided the initial condition and the input are smooth. For this class, the following balance equation holds, see (12.29)

$$\begin{aligned}\frac{d}{dt}\int_a^b w(\zeta, t)\lambda(\zeta)\overline{w(\zeta, t)} d\zeta &= |\lambda(b) w(b, t)|^2 - |\lambda(a) w(a, t)|^2 \\ &= |u(t)|^2 - |y(t)|^2.\end{aligned}$$

By Proposition 13.1.4 we conclude that the system is well-posed on the energy space. Since λ is strictly positive, we have that the energy norm $\int_a^b |w(\zeta, t)|^2 \lambda(\zeta) d\zeta$ is equivalent to the $L^2(a, b)$-norm, and so the system is also well-posed on $L^2(a, b)$.

Example 12.2.2 shows that the transfer function of (13.56)–(13.58) is given by $G(s) = e^{-p(b)s}$ with $p(b) > 0$. Thus the system is regular and property (13.59) follows immediately. □

For a p.d.e. with negative coefficient, we obtain a similar result.

Lemma 13.2.7. *Let θ be a negative continuous function on the interval $[a, b]$. We consider the system*

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta}\left(\theta(\zeta) w(\zeta, t)\right), \qquad w(\zeta, 0) = w_0(\zeta), \qquad \zeta \in [a, b]. \tag{13.60}$$

The value at a we choose as input

$$u(t) = \theta(a) w(a, t) \tag{13.61}$$

and as output we choose the value on the other end

$$y(t) = \theta(b) w(b, t). \tag{13.62}$$

The port-Hamiltonian system (13.60)–(13.62) *is regular on the state space $L^2(a, b)$. Its transfer function is given by*

$$G(s) = e^{n(b)s}, \tag{13.63}$$

where

$$n(\zeta) = \int_a^\zeta \theta(z)^{-1} dz, \qquad \zeta \in [a, b]. \tag{13.64}$$

The transfer function satisfies

$$\lim_{\mathrm{Re}(s) \to \infty} G(s) = 0. \tag{13.65}$$

Lemma 13.2.7 will be proved in Exercise 13.7.

In Theorem 13.2.2 we asserted that under some weak conditions every port-Hamiltonian system is well-posed provided the corresponding homogeneous equation generates a strongly continuous semigroup. In Sections 13.3 and 13.4 we prove this assertion. However, we like to explain the main ideas of the proof by means of an example. We choose the wave equation of Example 13.2.4. Since S is invertible, well-posedness will not change if we perform a basis transformation, $\tilde{x} = Sx$, see Exercise 13.3. After this basis transformation, the p.d.e. (13.51) reads

$$\begin{aligned} \frac{\partial \tilde{x}}{\partial t}(\zeta, t) &= \frac{\partial}{\partial \zeta}\left(\Delta \tilde{x}\right)(\zeta, t) + S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta, t) \\ &= \frac{\partial}{\partial \zeta}\begin{bmatrix} \gamma(\zeta)\tilde{x}_1(\zeta, t) \\ -\gamma(\zeta)\tilde{x}_2(\zeta, t) \end{bmatrix} + S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta, t). \end{aligned} \tag{13.66}$$

Thus via a basis transformation we obtain a set of simple p.d.e.'s, just two simple transport equations, but they are corrupted by the term $S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta,t)$. We first assume that this term is not present, and so we study the well-posedness of the collection of transport equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \tilde{x}_1(\zeta,t) \\ \tilde{x}_2(\zeta,t) \end{bmatrix} = \frac{\partial}{\partial \zeta}\begin{bmatrix} \gamma(\zeta)\tilde{x}_1(\zeta,t) \\ -\gamma(\zeta)\tilde{x}_2(\zeta,t) \end{bmatrix}. \tag{13.67}$$

Although it seems that these two p.d.e.'s are uncoupled, they are coupled via the boundary conditions. Ignoring the coupling via the boundary condition, we can apply Lemmas 13.2.6 and 13.2.7 directly. In Section 13.3 we investigate when the p.d.e. (13.67) with control and observation at the boundary is well-posed. In Section 13.4 we return to the original p.d.e., and apply Lemma 13.1.14 to show that ignoring bounded terms, like we did in (13.67) does not influence the well-posedness of the system. Since a basis transformation does not effect it either, we have proved Theorem 13.2.2 for the wave equation.

13.3 $P_1\mathcal{H}$ diagonal

In this section, we prove Theorem 13.2.2 in the situation that $P_1\mathcal{H}$ is diagonal, i.e., when $S = I$. Thus we consider the following diagonal port-Hamiltonian system

$$\frac{\partial}{\partial t}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix} = \frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix}\begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix}\right), \tag{13.68}$$

where for every $\zeta \in [a,b]$, $\Lambda(\zeta)$ is a diagonal (real) matrix, with positive numbers on the diagonal, and $\Theta(\zeta)$ is a diagonal (real) matrix, with negative numbers on the diagonal. Furthermore, we assume that Λ and Θ are continuously differentiable and that $\begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix}$ is an $n\times n$-matrix. We note that (13.68) is a port-Hamiltonian system with $\mathcal{H} := \begin{bmatrix} \Lambda & 0 \\ 0 & -\Theta \end{bmatrix}$ and $P_1 := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

The following boundary control and observation are of interest

$$u_s(t) := \begin{bmatrix} \Lambda(b)x_+(b,t) \\ \Theta(a)x_-(a,t) \end{bmatrix}, \tag{13.69}$$

$$y_s(t) := \begin{bmatrix} \Lambda(a)x_+(a,t) \\ \Theta(b)x_-(b,t) \end{bmatrix}. \tag{13.70}$$

Theorem 13.3.1. *Consider the p.d.e.* (13.68) *with boundary control* u_s *and boundary observation* y_s *as defined in* (13.69) *and* (13.70), *respectively.*

- *The system defined by* (13.68)–(13.70) *is well-posed and regular. Furthermore, the corresponding transfer function* G_s *converges to zero for* $\mathrm{Re}(s) \to \infty$.

- *We equip the port-Hamiltonian system* (13.68) *with a new set of inputs and outputs. The new input* $u(t)$ *is of the form*

$$u(t) = Ku_s(t) + Qy_s(t), \tag{13.71}$$

where K *and* Q *are two square* $n \times n$*-matrices with* $\begin{bmatrix} K & Q \end{bmatrix}$ *of rank* n*. The new output is of the form*

$$y(t) = O_1 u_s(t) + O_2 y_s(t), \tag{13.72}$$

where O_1 *and* O_2 *are* $k \times n$*-matrices. Concerning system* (13.68) *with input* u *and output* y*, we have the following results:*

1. *If* K *is invertible, then the system* (13.68), (13.71), *and* (13.72) *is well-posed and regular. Furthermore, the corresponding transfer function converges to* $O_1 K^{-1}$ *for* $\mathrm{Re}(s) \to \infty$
2. *If* K *is not invertible, then the operator* A_K *defined as*

$$A_K \begin{bmatrix} g_+(\zeta) \\ g_-(\zeta) \end{bmatrix} = \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix} \begin{bmatrix} g_+(\zeta) \\ g_-(\zeta) \end{bmatrix} \right) \tag{13.73}$$

with domain

$$D(A_K) = \left\{ \begin{bmatrix} g_+(\zeta) \\ g_-(\zeta) \end{bmatrix} \in H^1([a,b], \mathbb{K}^n) \middle| \right. \tag{13.74}$$
$$\left. K \begin{bmatrix} \Lambda(b) g_+(b) \\ \Theta(a) g_-(a) \end{bmatrix} + Q \begin{bmatrix} \Lambda(a) g_+(a) \\ \Theta(b) g_-(b) \end{bmatrix} = 0 \right\}$$

does not generate a C_0*-semigroup on* $L^2([a,b]; \mathbb{K}^n)$*.*

Note that part 2 implies that the homogeneous p.d.e. does not have a well-defined solution, when K is not invertible.

Proof. The first part is a direct consequence of Lemma 13.2.6 and 13.2.7 by noticing that the system (13.68)–(13.70) is built out of copies of the system (13.56)–(13.58) and the system (13.60)–(13.62). Furthermore, these sub-systems do not interact with each other. Thus in particular, the corresponding transfer function $G_s(s)$ is diagonal.

For the proof of the first part of the second assertion, with K invertible, we rewrite the new input, as $u_s(t) = K^{-1}u(t) - K^{-1}Qy_s(t)$, $t \geq 0$. This can be seen as a feedback interconnection on the system (13.68)–(13.70), as is depicted in Figure 13.2. Note that the system (13.68)–(13.70) is denoted by $\mathfrak{S}_s$. The system contains one feedback loop with feedback $K^{-1}Q$. By Theorem 13.1.12, we have that, if $I + G_s(s)K^{-1}Q$ is invertible for some complex s and if this inverse exists and is bounded on a right half-plane, then the closed loop system is well-posed. Since $\lim_{\mathrm{Re}(s)\to\infty} G_s(s) = 0$, this holds for every K^{-1} and Q. Thus under the

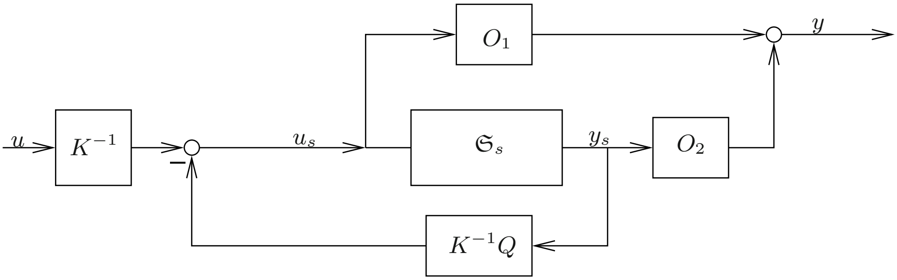

Figure 13.2: The system (13.68) with input (13.71) and output (13.72)

assumption that K is invertible, the system (13.68) with input and output given by (13.71) and (13.72) is well-posed. The regularity follows easily. By regarding the loops in Figure 13.2, we see that the feed-through term is given by $O_1 K^{-1}$.

It remains to show that A_K does not generate a C_0-semigroup if K is not invertible. Since K is singular, there exists a non-zero $v \in \mathbb{K}^n$ such that $v^* K = 0$. Since $\begin{bmatrix} K & Q \end{bmatrix}$ has full rank, we obtain that $q^* := v^* Q \neq 0$. This implies that at least one of the components of the vector q is unequal to zero. For the sake of the argument, we assume that this holds for the first component.

Assume that A_K is the infinitesimal generator of a C_0-semigroup. This implies that for every $x_0 \in D(A_K)$ the abstract differential equation

$$\dot{x}(t) = A_K x(t), \qquad x(0) = x_0 \tag{13.75}$$

has a classical solution, i.e., for every $x_0 \in D(A_K)$ there exists a function $x(\zeta, t) := \begin{bmatrix} x_+(\zeta,t) \\ x_-(\zeta,t) \end{bmatrix}$ which is a classical solution to the p.d.e. (13.68) with initial condition $x(\cdot, 0) = x_0$, and satisfies for all $t \geq 0$ the boundary condition

$$K \begin{bmatrix} \Lambda(b) x_+(b,t) \\ \Theta(a) x_-(a,t) \end{bmatrix} + Q \begin{bmatrix} \Lambda(a) x_+(a,t) \\ \Theta(b) x_-(b,t) \end{bmatrix} = 0.$$

Using the vectors v and q, the function x satisfies

$$0 = q^* \begin{bmatrix} \Lambda(a) x_+(a,t) \\ \Theta(b) x_-(b,t) \end{bmatrix}, \quad t \geq 0. \tag{13.76}$$

Now we construct an initial condition in $D(A_K)$, for which this equality does not hold. Note that we have chosen the first component of q unequal to zero.

The initial condition x_0 is chosen to have all components zero except for the first one. For this first component we choose an arbitrary function in $H^1(a,b)$ which is zero at a and b, but nonzero everywhere else on the open set (a,b). Clearly this initial condition is an element of the domain of A_K. Now we solve (13.68).

Standard p.d.e. theory implies that the solution of (13.68) can be written as

$$x_{+,m}(\zeta,t) = f_{+,m}(p_m(\zeta)+t)\lambda_m(\zeta)^{-1}, \tag{13.77}$$
$$x_{-,\ell}(\zeta,t) = f_{-,\ell}(n_\ell(\zeta)+t)\theta_\ell(\zeta)^{-1}, \tag{13.78}$$

where λ_m and θ_ℓ are the m-th and the ℓ-th diagonal element of Λ and Θ, respectively. Furthermore, $p_m(\zeta) = \int_a^\zeta \lambda_m(\zeta)^{-1}d\zeta$, $n_\ell(\zeta) = \int_a^\zeta \theta_\ell(\zeta)^{-1}d\zeta$, see also Exercises 13.6 and 13.7. In particular, we have $p_m(a) = n_\ell(a) = 0$. The functions $f_{+,m}, f_{-,\ell}$ need to be determined from the boundary and initial conditions.

Using the initial condition, we obtain $f_{+,m}(p_m(\zeta)) = \lambda_m(\zeta)x_{0,+,m}(\zeta)$ and $f_{-,\ell}(n_\ell(\zeta)) = \theta_\ell(\zeta)x_{0,-,\ell}(\zeta)$. Since $p_m > 0$, and $n_\ell < 0$, the initial condition determines $f_{+,m}$ on a (small) positive interval, and $f_{-,\ell}$ on a small negative interval. By our choice of the initial condition, we find that

$$\begin{aligned} f_{+,1}(\zeta) &= \lambda_1(p_1^{-1}(\zeta))x_{0,+,1}(p_1^{-1}(\zeta)), && \zeta\in[0,p_1(b)),\\ f_{+,m}(\zeta) &= 0, && \zeta\in[0,p_m(b)), \quad m\ge 2,\\ f_{-,\ell}(\zeta) &= 0, && \zeta\in[n_\ell(b),0), \quad \ell\ge 1. \end{aligned} \tag{13.79}$$

The solution $x(\zeta,t)$ must also satisfy (13.76), thus for every $t>0$ we have

$$0 = q^* \begin{bmatrix} f_{+,1}(t) \\ \vdots \\ f_{-,1}(n_1(b)+t) \\ \vdots \end{bmatrix}. \tag{13.80}$$

Combining this with (13.79), we find

$$0 = q_1 f_{+,1}(p_1(\zeta)) = q_1 x_{0,+,1}(\zeta)\lambda_1(\zeta)$$

on some interval $[a,\beta]$. Since q_1 and λ_1 are unequal to zero, we find that $x_{0,+,1}$ must be zero on some interval. This is in contradiction to our choice of the initial condition. Thus A_K cannot be the infinitesimal generator of a C_0-semigroup. □

13.4 Proof of Theorem 13.2.2

In this section we use the results of the previous section to prove Theorem 13.2.2.

By Assumption 1 of Theorem 13.2.2, the matrices P_1 and $\mathcal{H}$ satisfy the equation (13.50):

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta).$$

We introduce the new state vector

$$\tilde{x}(\zeta,t) = S(\zeta)x(\zeta,t), \qquad \zeta\in[a,b]. \tag{13.81}$$

Under this basis transformation, the p.d.e. (13.45) becomes

$$\frac{\partial \tilde{x}}{\partial t}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(\Delta \tilde{x}\right)(\zeta,t) + S(\zeta)\frac{dS^{-1}(\zeta)}{d\zeta}\Delta(\zeta)\tilde{x}(\zeta,t) \\ + S(\zeta)P_0(\zeta)S(\zeta)^{-1}\tilde{x}(\zeta,t), \qquad \tilde{x}_0(\zeta) := \tilde{x}(\zeta,0) = S(\zeta)x_0(\zeta). \quad (13.82)$$

The equations (13.46)–(13.48) imply the existence of matrices M_{11}, M_{12}, M_{21}, M_{22}, $\tilde{M}_{11}$, $\tilde{M}_{12}$, $\tilde{M}_{21}$, $\tilde{M}_{22}$, C_1, C_2, $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\begin{aligned} 0 &= M_{11}P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(b,t) + M_{12}P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(a,t) \\ &= \tilde{M}_{11}\Delta(b)\tilde{x}(b,t) + \tilde{M}_{12}\Delta(a)\tilde{x}(a,t), && (13.83)\\ u(t) &= M_{21}P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(b,t) + M_{22}P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(a,t) \\ &= \tilde{M}_{21}\Delta(b)\tilde{x}(b,t) + \tilde{M}_{22}\Delta(a)\tilde{x}(a,t), && (13.84)\\ y(t) &= C_1P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(b,t) + C_2P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(a,t) \\ &= \tilde{C}_1\Delta(b)\tilde{x}(b,t) + \tilde{C}_2\Delta(a)\tilde{x}(a,t). && (13.85) \end{aligned}$$

We introduce $\tilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$. We have

$$\begin{bmatrix} \tilde{M}_{j1} & \tilde{M}_{j2} \end{bmatrix} = \begin{bmatrix} M_{j1} & M_{j2} \end{bmatrix} \begin{bmatrix} P_1^{-1}S(b)^{-1} & 0 \\ 0 & P_1^{-1}S(a)^{-1} \end{bmatrix}, \quad j = 1,2$$

and

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} P_1^{-1}S(b)^{-1} & 0 \\ 0 & P_1^{-1}S(a)^{-1} \end{bmatrix}.$$

Since the matrix $\begin{bmatrix} P_1^{-1}S(b)^{-1} & 0 \\ 0 & P_1^{-1}S(a)^{-1} \end{bmatrix}$ has full rank, the rank conditions in Theorem 13.2.2 imply similar rank conditions for $\tilde{M}$ and $\tilde{C}$.

Using Lemma 13.1.14, we only have to prove the result for the p.d.e.

$$\frac{\partial \tilde{x}}{\partial t}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(\Delta \tilde{x}\right)(\zeta,t) \quad (13.86)$$

with boundary conditions, inputs, and outputs as described in (13.83)–(13.85).

If condition (13.83) is not present, then Theorem 13.3.1 implies that the above system is well-posed and regular if and only if the homogeneous p.d.e. generates a C_0-semigroup on $L^2([a,b];\mathbb{K}^n)$. Since the state transformation (13.81) defines a bounded mapping on $L^2([a,b];\mathbb{K}^n)$, we have proved Theorem 13.2.2 provided there is no condition of the form (13.47).

Thus it remains to prove Theorem 13.2.2 in the case that we have set part of the boundary conditions to zero. Or equivalently, to prove that the system (13.83)–(13.86) is well-posed and regular if and only if the homogeneous p.d.e. generates a C_0-semigroup.

We replace (13.83) by

$$v(t) = \tilde{M}_{11}\Delta(b)\tilde{x}(b,t) + \tilde{M}_{12}\Delta(a)\tilde{x}(a,t), \tag{13.87}$$

where we regard v as a new input. Hence we have the system (13.86) with the new extended input

$$\begin{bmatrix} v(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \tilde{M}_{11} \\ \tilde{M}_{21} \end{bmatrix} \Delta(b)\tilde{x}(t,b) + \begin{bmatrix} \tilde{M}_{12} \\ \tilde{M}_{22} \end{bmatrix} \Delta(a)\tilde{x}(t,a) \tag{13.88}$$

and the output (13.85). Thus we obtain a system without a condition of the form (13.83). For the new system we know that it is well-posed and regular if and only if the homogeneous equation generates a C_0-semigroup.

Assume that the system (13.86), (13.88) and (13.85) is well-posed, then we may choose any (locally) square integrable input. In particular, we may choose $v \equiv 0$. Thus the system (13.83)–(13.86) is well-posed and regular as well.

Assume next that the p.d.e. with the extended input in (13.88) set to zero does not generate a C_0-semigroup. Since this gives the same homogeneous p.d.e. as (13.86) with (13.83) and u in (13.84) set to zero, we know that this p.d.e. does not generate a C_0-semigroup either. This finally proves Theorem 13.2.2. □

13.5 Well-posedness of the vibrating string

In this section we illustrate the usefulness of Theorem 13.2.2 by applying it to the vibrating string of Example 7.1.1.

By Example 13.2.4 for the vibrating string we have

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}.$$

We want to illustrate the theory and proofs derived in the previous sections, and therefore we do not directly check if for a (to-be-given) set of boundary conditions that the semigroup condition is satisfied. Instead of that, we rewrite the system in its diagonal form, and check the conditions of Theorem 13.3.1. As we have seen in Section 13.4, the proof of Theorem 13.2.2 follows after a basis transformation directly from Theorem 13.3.1. Hence we first diagonalize $P_1\mathcal{H}$. Although all the results hardly change, we assume for simplicity of notation that Young's modulus T and the mass density ρ are constant. Being physical constants, they are naturally positive.

From equation (13.54) we know that the operator $P_1\mathcal{H}$ is diagonalizable:

$$P_1\mathcal{H} = S^{-1}\Delta S = \begin{bmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{bmatrix}, \tag{13.89}$$

where γ is positive and satisfies $\gamma^2 = \frac{T}{\rho}$.

Hence the state transformation under which the p.d.e. becomes diagonal is

$$\tilde{x} = \frac{1}{2}\begin{bmatrix} \frac{1}{\gamma} & \rho \\ -\frac{1}{\gamma} & \rho \end{bmatrix} x.$$

Since we assumed that $\gamma > 0$, the functions $\tilde{x}_1$ and $\tilde{x}_2$ correspond to x_+ and x_- in equation (13.68), respectively and Λ, Θ to γ and $-\gamma$, respectively. Hence the input and output u_s and y_s defined for the diagonal system (13.68) by the equations (13.69)–(13.70) are expressed in the original coordinates by

$$u_s(t) = \frac{1}{2}\begin{bmatrix} x_1(b,t) + \gamma\rho x_2(b,t) \\ x_1(a,t) - \gamma\rho x_2(a,t) \end{bmatrix}, \tag{13.90}$$

$$y_s(t) = \frac{1}{2}\begin{bmatrix} x_1(a,t) + \gamma\rho x_2(a,t) \\ x_1(b,t) - \gamma\rho x_2(b,t) \end{bmatrix}. \tag{13.91}$$

This pair of boundary input and output variables consists of complementary linear combinations of the momentum x_1 and the strain x_2 at the boundaries: however they lack an obvious physical interpretation. One could consider another choice of boundary input and outputs, for instance the velocity and the strain at the boundary points and choose as input

$$u_1(t) = \begin{bmatrix} \frac{x_1}{\rho}(b,t) \\ x_2(a,t) \end{bmatrix} \tag{13.92}$$

and as output

$$y_1(t) = \begin{bmatrix} \frac{x_1}{\rho}(a,t) \\ x_2(b,t) \end{bmatrix}. \tag{13.93}$$

We may apply Theorem 13.3.1 to check whether this system is well-posed, and to find the feed-through. Expressing the input-output pair (u_1, y_1) in terms of (u_s, y_s) gives

$$u_1(t) = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{bmatrix} u_s(t) + \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{bmatrix} y_s(t), \tag{13.94}$$

$$y_1(t) = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{bmatrix} u_s(t) + \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{bmatrix} y_s(t). \tag{13.95}$$

Hence

$$K = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{bmatrix}, \tag{13.96}$$

$$O_1 = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{bmatrix}, \qquad O_2 = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{bmatrix}. \tag{13.97}$$

Since K is invertible, the system with the input-output pair (u_1, y_1) is well-posed and regular, and the feed-through term is given by $O_1 K^{-1} = \begin{bmatrix} 0 & -\gamma \\ \frac{1}{\gamma} & 0 \end{bmatrix}$.

Since the states are defined as $x_1 = \rho \frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial \zeta}$, the control and observation are easily formulated using w. Namely, $u_1(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(t,b) \\ \frac{\partial w}{\partial \zeta}(t,a) \end{bmatrix}$ and $y_1(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(t,a) \\ \frac{\partial w}{\partial \zeta}(t,b) \end{bmatrix}$, respectively. Hence we observe the velocity and strain at opposite ends.

Next we show that if we control the velocity and strain at the same end, this does not give a well-posed system. The control and observation are given by

$$u_2(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(b,t) \\ \frac{\partial w}{\partial \zeta}(b,t) \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\rho}(b,t) \\ x_2(b,t) \end{bmatrix} \tag{13.98}$$

and as output

$$y_2(t) = \begin{bmatrix} \frac{\partial w}{\partial t}(a,t) \\ \frac{\partial w}{\partial \zeta}(a,t) \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\rho}(a,t) \\ x_2(a,t) \end{bmatrix}. \tag{13.99}$$

It is easy to see that

$$u_2(t) = \begin{bmatrix} \frac{1}{\rho} & 0 \\ \frac{1}{\gamma\rho} & 0 \end{bmatrix} u_s(t) + \begin{bmatrix} 0 & \frac{1}{\rho} \\ 0 & -\frac{1}{\gamma\rho} \end{bmatrix} y_s(t). \tag{13.100}$$

Clearly the matrix in front of u_s is not invertible, and hence we conclude by Theorem 13.3.1 that the wave equation with the homogeneous boundary conditions $u_2 = 0$ does not generate a C_0-semigroup. Hence this system is not well-posed.

13.6 Exercises

13.1. In this exercise we show that a homogeneous port-Hamiltonian system needs not to generate a contraction semigroup. Consider the transport equation

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial x}{\partial \zeta}(\zeta,t), \quad x(\zeta,0) = x_0(\zeta), \qquad \zeta \in [0,1], \tag{13.101}$$

$$x(1,t) = 2x(0,t). \tag{13.102}$$

(a) Show that for an initial condition x_0 which is continuously differentiable and satisfies $x_0(1) = 2x_0(0)$ and $x_0'(1) = 2x_0'(0)$ the (classical) solution of (13.101)–(13.102) is given by

$$x(\zeta,t) = 2^n x_0(\tau), \tag{13.103}$$

where $\zeta + t = n + \tau$, $\tau \in [0,1]$, $n \in \mathbb{N} \cup \{0\}$.

(b) Show that the mapping $x_0 \mapsto x(\cdot,t)$ with $x(\zeta,t)$ given by (13.103) defines a C_0-semigroup on $L^2(0,1)$.

(c) Conclude that (13.103) is the mild solution of (13.101)–(13.102) for any initial condition $x_0 \in L^2(0,1)$.

13.2. Let X be a Hilbert space with norm, $\|\cdot\|_X$ and let $\|\cdot\|_n$ be an equivalent norm, i.e., there exists $\alpha, \beta > 0$ such that for all $x \in X$ there holds $\alpha\|x\|_X \leq \|x\|_n \leq \beta\|x\|_X$.

Show that if boundary control system (13.8)–(13.10) satisfying Assumption 13.1.2 is well-posed with respect to the state space norm $\|\cdot\|_X$, then the system is also well-posed with respect to the state space norm $\|\cdot\|_n$.

13.3. Consider the boundary control system (13.8)–(13.10) satisfying Assumption 13.1.2. Let S be a be linear operator in $\mathcal{L}(X)$ which is boundedly invertible. Introduce the new state variable $\tilde{x} = Sx$.

(a) Rewrite the boundary control system (13.8)–(13.10) as a boundary control system in the state variable $\tilde{x}$. Furthermore, show that this new boundary control system satisfies Assumption 13.1.2.

(b) Show that the transformed boundary control system is again well-posed when the original boundary control system is well-posed.

(c) Prove that both systems have the same transfer function.

13.4. Assume the the boundary control system (13.8)–(13.10) is well-posed and define

$$y_{\text{new}} := O_1 u(t) + O_2 y(t), \tag{13.104}$$

where O_1 and O_2 are bounded linear operators. Show the port-Hamiltonian systems remains well-posed if we replace (13.10) by (13.104).

13.5. Use Theorem 11.2.1 to prove Lemma 13.1.6.

13.6. In this exercise we prove some further results concerning the system (13.56)–(13.58).

(a) Show that the system (13.56)–(13.58) is a port-Hamiltonian system of the form (13.45)–(13.48).

(b) Show that the solution of (13.56)–(13.57) is given by

$$w(\zeta, t) = f(p(\zeta) + t)\lambda(\zeta)^{-1}, \tag{13.105}$$

where

$$p(\zeta) = \int_a^\zeta \lambda(z)^{-1} dz, \qquad \zeta \in [a, b], \tag{13.106}$$

$$f(p(\zeta)) = \lambda(\zeta) w_0(\zeta), \qquad \zeta \in [a, b], \tag{13.107}$$

$$f(p(b) + t) = u(t), \qquad t > 0. \tag{13.108}$$

13.7. In this exercise we prove Lemma 13.2.7.

(a) Show that the solution of (13.60)–(13.61) is given as

$$w(\zeta,t) = f(n(\zeta)+t)\theta(\zeta)^{-1}, \tag{13.109}$$

where

$$n(\zeta) = \int_a^\zeta \theta(z)^{-1} dz, \tag{13.110}$$

$$f(n(\zeta)) = \theta(\zeta) w_0(\zeta), \qquad z \in [a,b], \tag{13.111}$$

$$f(t) = u(t), \qquad t > 0. \tag{13.112}$$

(b) Use part a to prove Lemma 13.2.7.

13.7 Notes and references

We have defined well-posedness for boundary control systems. However, it is also possible to define well-posedness more abstractly. In order to do this we start with a bounded linear operator-valued function $(\mathcal{S}(t))_{t\geq 0} = \left(\begin{bmatrix} \mathcal{S}_{11}(t) & \mathcal{S}_{12}(t) \\ \mathcal{S}_{21}(t) & \mathcal{S}_{22}(t) \end{bmatrix} \right)_{t\geq 0}$, satisfying properties (13.27), (13.33)–(13.35) and $(\mathcal{S}_{11}(t))_{t\geq 0}$ is a C_0-semigroup, see the book of Staffans [51].

Our class of port-Hamiltonian systems is most likely the only class of p.d.e.'s which allows for such a simple characterization of well-posedness. For a general p.d.e. it is usually very hard to prove well-posedness.

The first condition of Theorem 13.2.2 implies that $P_1\mathcal{H}$ is diagonalizable via a continuously differentiable basis transformation. Since $S(\zeta)$ in equation (13.50) contains the eigenvectors, we need the eigenvectors to be continuously differentiable. Example II.5.3 of Kato [30] gives a symmetric matrix-valued function which is infinitely continuously differentiable, but the eigenvectors are not even once continuously differentiable. If $P_1\mathcal{H}$ is a C^1-function such that for all $\zeta \in [a,b]$, the eigenvalues $P_1\mathcal{H}(\zeta)$ are simple, then the eigenvalues and eigenvectors can be chosen continuously differentiable, see Kato [30, Theorem II.5.4].

In our proof of Theorem 13.2.2 we only use that Δ is continuous. However, when $P_1\mathcal{H}$ is continuously differentiable, then so are its eigenvalues. This follows from the fact that the eigenvalues of $P_1\mathcal{H}$ and $\mathcal{H}^{\frac{1}{2}}P_1\mathcal{H}^{\frac{1}{2}}$ are the same, and Theorem II.6.8 of [30].

From the proof of Theorem 13.2.2 follows that the operator A corresponding to the homogeneous equation generates a C_0-semigroup if and only if the matrix K is invertible, see Theorem 13.3.1. Since the matrix K is obtained after a basis transformation, and depends on the negative and positive eigenvalues of $P_1\mathcal{H}$, it is not easy to rewrite this condition in terms of the matrix W_B.

A semigroup can be extended to a C_0-group, if for every initial condition the homogeneous p.d.e. has a solution for negative time. Using once more the proof of Theorem 13.3.1, the operator A of Theorem 13.2.2 generates a C_0-group if and only if K and Q are invertible matrices.

Theorem 13.1.12 was first proved by Weiss [60]. Moreover, we note that the content of this chapter is based on the paper by Zwart et al., [65].

Appendix A

Integration and Hardy Spaces

A.1 Integration theory

In this section, we wish to extend the ideas of Lebesgue integration of complex-valued functions to vector-valued and operator-valued functions, which take their values in a separable Hilbert space X or in the Banach space $\mathcal{L}(X_1, X_2)$, where X_1, X_2 are separable Hilbert spaces. By $\mathcal{L}(X_1, X_2)$ we denote the class of all bounded, linear operators from X_1 to X_2. As main references, we have used Arendt, Batty, Hieber and Neubrander [3], Diestel and Uhl [12], Dunford and Schwartz [13], and Hille and Phillips [24].

Throughout this section, we use the notation Ω for a closed subset of $\mathbb{R}$, and $(\Omega, \mathcal{B}, \lambda)$ for the measure space with the Lebesgue measure λ and (Lebesgue) measurable subsets $\mathcal{B}$. It is possible to develop a Lebesgue integration theory based on various measurability concepts.

Definition A.1.1. Let $\mathcal{W}$ be a Banach space. A function $f : \Omega \to \mathcal{W}$ is called *simple* if there exist $w_1, w_2, \ldots, w_n \in \mathcal{W}$ and $E_1, E_2, \ldots, E_n \in \mathcal{B}$ such that $f = \sum_{i=1}^{n} w_i \ {}_{E_i}$, where ${}_{E_i}(t) = 1$ if $t \in E_i$ and 0 otherwise.

Let X_1, X_2 be two separable Hilbert spaces, and let $F : \Omega \to \mathcal{L}(X_1, X_2)$ and $f : \Omega \to X_1$.

1. F is *uniformly (Lebesgue) measurable* if there exists a sequence of simple functions $F_n : \Omega \to \mathcal{L}(X_1, X_2)$ such that

$$\lim_{n\to\infty} F_n(t) = F(t) \text{ for almost all } t \in \Omega.$$

2. f is *strongly (Lebesgue) measurable* if there exists a sequence of simple functions $f_n : \Omega \to X_1$ such that

$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t \in \Omega.$$

   F is *strongly measurable* if Fx_1 is strongly measurable for every $x_1 \in X_1$.

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1,

3. f is *weakly (Lebesgue) measurable* if $\langle f, x_1 \rangle$ is measurable for every $x_1 \in X_1$.

   F is *weakly measurable* if Fx_1 is weakly measurable for every $x_1 \in X_1$.

It is easy to see that uniform measurability implies strong measurability, which implies weak measurability. For our separable Hilbert spaces X_1 and X_2, the concepts weak and strong measurability coalesce.

Lemma A.1.2. *For the case that X_1 and X_2 are separable Hilbert spaces the concepts of weak and strong measurability in Definition* A.1.1 *coincide.*

Proof. See Hille and Phillips [24, theorem 3.5.3] or Yosida [61, theorem in Section V.4]. □

We often consider the inner product of two weakly measurable functions.

Lemma A.1.3. *Let X be a separable Hilbert space, and let $f_1, f_2 : \Omega \to X$ be two weakly measurable functions. The complex-valued function $\langle f_1(\cdot), f_2(\cdot) \rangle$ defined by the inner product of these functions is a measurable function.*

Proof. This follows directly from Lemma A.1.2 and Definition A.1.1. □

Lemma A.1.4. *Let $f : \Omega \to X_1$ and $F : \Omega \to \mathcal{L}(X_1, X_2)$ be weakly measurable, then $\|f\|$ and $\|F\|$ are measurable, scalar-valued functions.*

Proof. The first assertion follows directly from the previous lemma, since $\|f\| = \sqrt{\langle f(\cdot), f(\cdot) \rangle}$. Since X_1 and X_2 are separable, there exist countable dense subsets Z_1 and Z_2 of X_1 and X_2, respectively. Since these sets are dense, we obtain

$$\|F(t)\| = \sup_{\substack{z_1 \in Z_1,\ z_1 \neq 0 \\ z_2 \in Z_2,\ z_2 \neq 0}} \frac{|\langle F(t)z_1, z_2 \rangle|}{\|z_1\|\|z_2\|}, \quad t \in \Omega.$$

By assumption the functions $\frac{|\langle F(t)z_1, z_2 \rangle|}{\|z_1\|\|z_2\|}$ are measurable, and since a (countable) supremum of measurable functions is measurable, we conclude that $\|F(\cdot)\|$ is measurable. □

The notion of the Lebesgue integral follows naturally from the measurability concepts given in Definition A.1.1.

Definition A.1.5. Suppose that $(\Omega, \mathcal{B}, \lambda)$ is the Lebesgue measure space and that $E \in \mathcal{B}$.

1. Let $\mathcal{W}$ be a Banach space and let $f : \Omega \to \mathcal{W}$ be a simple function given by $f = \sum_{i=1}^n w_i \; {}_{E_i}$, where the E_i are disjoint. We define f to be *Lebesgue integrable* over E if $\|f\|$ is Lebesgue integrable over E, that is, $\sum_{i=1}^n \|w_i\| \lambda(E_i \cap E) < \infty$, where $\lambda(\cdot)$ denotes the Lebesgue measure of the set and we follow the usual convention that $0 \cdot \infty = 0$. The *Lebesgue integral* of f over E is given by $\sum_{i=1}^n w_i \lambda(E_i \cap E)$ and will be denoted by $\int_E f(t)dt$.

2. Let X_1 and X_2 be two separable Hilbert spaces. The uniformly measurable function $F : \Omega \to \mathcal{L}(X_1, X_2)$ is *Lebesgue integrable* over E if there exists a sequence of simple integrable functions F_n converging almost everywhere to F and such that
$$\lim_{n\to\infty} \int_E \|F(t) - F_n(t)\|_{\mathcal{L}(X_1,X_2)} dt = 0.$$
We define the *Lebesgue integral* by
$$\int_E F(t)dt = \lim_{n\to\infty} \int_E F_n(t)dt.$$

3. Let X be a separable Hilbert space. The strongly measurable function $f : \Omega \to X$ is *Lebesgue integrable* over E if there exists a sequence of simple integrable functions f_n converging almost everywhere to f and such that
$$\lim_{n\to\infty} \int_E \|f(t) - f_n(t)\|_X dt = 0.$$
We define the *Lebesgue integral* by
$$\int_E f(t)dt = \lim_{n\to\infty} \int_E f_n(t)dt.$$

The integrals in the above definition are also called *Bochner integrals* in the literature. If $f : \Omega \to X$ is Lebesgue integrable over E, then $\int_E f(t)dt$ is an element of X. Similarly, if $f : \Omega \to \mathcal{L}(X_1, X_2)$ is Lebesgue integrable over E, then $\int_E F(t)dt \in \mathcal{L}(X_1, X_2)$. For functions from $\mathbb{R}$ to a separable Hilbert space X, there is a simple criterion to test whether a function is Lebesgue integrable.

Lemma A.1.6. *Let $f : \Omega \to X$, where X is a separable Hilbert space. f is Lebesgue integrable over $E \in \mathcal{B}$ if and only if the function $\langle x, f(\cdot)\rangle$ is measurable for every $x \in X$ and $\int_E \|f(t)\|dt < \infty$.*

Proof. See Arendt, Batty, Hieber and Neubrander [3, Theorem 1.1.4], or Hille and Phillips [24, theorem 3.7.4], noting that weak and strong measurability are the same for separable Hilbert spaces (Lemma A.1.2). □

Proposition A.1.7. *Let X be a separable Hilbert space, let $f : [a, b] \to X$ be Lebesgue integrable and $F(t) := \int_a^t f(s)\, ds$ for $t \in [a, b]$. Then F is differentiable a.e. and $F' = f$ a.e., that is, for almost all $t \in [a, b]$ we have $\lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = f(t)$.*

Proof. See Arendt, Batty, Hieber and Neubrander [3, Theorem 1.2.2]. □

In the case of operator-valued functions $F : \Omega \to \mathcal{L}(X_1, X_2)$, where X_1 and X_2 are separable Hilbert spaces, we need to distinguish between the Lebesgue integral $\int_E F(t)\, dt$ for the case that F is uniformly (Lebesgue) measurable and the Lebesgue integral $\int_E F(t)x\, dt$ for the case that F is only strongly (Lebesgue) measurable.

Example A.1.8. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a separable Hilbert space X. Since $(T(t))_{t\geq 0}$ is strongly continuous, it is strongly measurable. In fact, Hille and Phillips [24, theorem 10.2.1] show that the C_0-semigroup is uniformly measurable if and only if it is uniformly continuous. Now the only uniformly continuous semigroups are those whose infinitesimal generator (see Definition 5.2.1) is a bounded operator, Hille and Phillips [24, theorem 9.4.2], and so $(T(t))_{t\geq 0}$ will only be strongly measurable in general. Thus $\int_0^1 T(t)x\,dt$ is a well-defined Lebesgue integral for any $x \in X$, but $\int_0^1 T(t)\,dt$ is in general not well-defined.

Next we study whether $\int_0^\tau T(\tau - s)F(s)ds$ has a meaning.

Example A.1.9. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a separable Hilbert space X, and let $F : [0, \infty) \to \mathcal{L}(U, X)$ be weakly measurable, U is a Hilbert space, and $\|F\| \in L^1(0, \tau)$. Since $(T^*(t))_{t\geq 0}$ is also a C_0-semigroup, $T^*(\cdot)x$ is continuous, see [10, Theorem 2.2.6], and so strongly measurable. Furthermore, by definition, we have that $F(\cdot)u$ is weakly measurable. Hence Lemma A.1.3 shows that $\langle x, T(\tau - \cdot)F(\cdot)u\rangle = \langle T^*(\tau - \cdot)x, F(\cdot)u\rangle$ is measurable for all $x \in X$, $u \in U$. So from Lemma A.1.6 we have that for each $u \in U$ $\int_0^\tau T(\tau - s)F(s)uds$ is a well-defined Lebesgue integral. However, $\int_0^\tau T(\tau - s)F(s)ds$ need not be a well-defined Lebesgue integral, since the integrand will not to be uniformly measurable in general.

If an operator-valued function is not uniformly measurable, but only weakly measurable, there is still the possibility to define the so-called Pettis integral.

Definition A.1.10. Let X_1 and X_2 be separable Hilbert spaces and let $F : \Omega \to \mathcal{L}(X_1, X_2)$. If for all $x_1 \in X_1$ and $x_2 \in X_2$ we have that the function $\langle x_2, F(\cdot)x_1\rangle \in L^1(\Omega)$, then we say that F is *Pettis integrable*. Furthermore, for all $E \in \mathcal{B}$, we call $\int_E F(t)dt$ defined by

$$\langle x_2, \int_E F(t)dtx_1\rangle := \int_E \langle x_2, F(t)x_1\rangle dt \tag{A.1}$$

the *Pettis integral* of F over E and $\int_E F(t)x_1dt$ the *Pettis integral* of $F(\cdot)x_1$ over E.

As for the Lebesgue integral we have that the Pettis integral $\int_E F(t)dt$ is an element of $\mathcal{L}(X_1, X_2)$. It also has the usual properties such as linearity

$$\int_E (\alpha F_1(t) + \beta F_2(t))\,dt = \alpha \int_E F_1(t)dt + \beta \int_E F_2(t)dt. \tag{A.2}$$

From the definition of the Pettis integral, we see that a weakly measurable function $F : \Omega \to \mathcal{L}(X_1, X_2)$ is Pettis integrable if and only if

$$\int_E |\langle x_2, F(t)x_1\rangle|dt < \infty. \tag{A.3}$$

In particular, if this weakly measurable function satisfies $\int_E \|F(t)\|dt < \infty$, then the condition (A.3) is satisfied and so it is Pettis integrable. Furthermore, it is easy to see that if F is an integrable simple function, then the Pettis integral equals the Lebesgue integral. From the definition of the Lebesgue integral, it follows easily that if the Lebesgue integral of a function exists, then the Pettis integral also exists, and they are equal.

In the following examples, we re-examine Examples A.1.8 and A.1.9, which we considered as Lebesgue integrals.

Example A.1.11. We recall from Example A.1.8 that the C_0-semigroup $(T(t))_{t\geq 0}$ on the separable Hilbert space X is in general only strongly measurable and so while $\int_0^1 T(t)xdt$ exists as a Lebesgue integral $\int_0^1 T(t)dt$ does in general not. We show next that it does exist as a Pettis integral. Since $(T(t))_{t\geq 0}$ is strongly continuous, we have that $\langle x_1, T(t)x_2\rangle$ is measurable for every $x_1, x_2 \in X$. From Theorem 5.1.5 we have that $\int_0^1 \|T(t)\|dt < \infty$. Thus by Definition A.1.10 the Pettis integral $\int_0^1 T(t)dt$ is well-defined. If the infinitesimal generator A of $(T(t))_{t\geq 0}$ is invertible, then using Theorem 5.2.2 we can even calculate this Pettis integral to obtain

$$\int_0^1 T(t)dt = A^{-1}T(1) - A^{-1}.$$

Example A.1.12. From Example A.1.9 we recall that $\int_0^\tau T(\tau - s)F(s)ds$ is in general not a well-defined Lebesgue integral. We already showed that $\langle x, T(\tau - \cdot) F(\cdot)u\rangle$ is Lebesgue measurable for all $x \in X$, $u \in U$. Furthermore, we see that

$$\int_0^\tau \|T(\tau - s)F(s)\|ds \leq M_\omega e^{\omega\tau} \int_0^\tau \|F(s)\|ds < \infty.$$

So by Definition A.1.10 the integrals $\int_0^\tau T(\tau - s)F(s)ds$ and $\int_0^\tau T(\tau - s)F(s)uds$ are well-defined as Pettis integrals, where only the latter one is well-defined as a Lebesgue integral.

Most of the integrals we use in this lecture notes satisfy the conditions in Lemma A.1.6.

Example A.1.13. Consider $\int_0^\tau T(\tau - s)Bu(s)ds$, where $(T(t))_{t\geq 0}$ is a C_0-semigroup on a separable Hilbert space X, $B \in \mathcal{L}(U, X)$, U is a separable Hilbert space and $u \in L^1([0, \tau]; U)$ (see Definition A.1.14). Then, as in Example A.1.12, $\langle x, T(\tau - \cdot)Bu(\cdot)\rangle$ is measurable for all $x \in X$ and $\int_0^\tau \|T(\tau - s)Bu(s)\|ds \leq M_\omega e^{\omega\tau}\|B\| \int_0^\tau \|u(s)\|ds < \infty$. So by Lemma A.1.6, the integral is well-defined as a Pettis or as a Lebesgue integral.

To avoid confusion between the Pettis and Lebesgue integrals we introduce the following notation.

Definition A.1.14. Let X_1, X_2, and X be separable Hilbert spaces, and let Ω be a closed subset of $\mathbb{R}$. We define the following spaces:

$$\begin{aligned} P(\Omega;\mathcal{L}(X_1,X_2)) &:= \{F:\Omega\to\mathcal{L}(X_1,X_2) \mid \langle x_2, F(\cdot)x_1\rangle \text{ is measurable} \\ &\qquad \text{for every } x_1\in X_1 \text{ and } x_2\in X_2\}. \\ P^p(\Omega;\mathcal{L}(X_1,X_2)) &:= \{F\in P(\Omega;\mathcal{L}(X_1,X_2)) \mid \|F\|_p := \\ &\qquad \left(\int_\Omega \|F(t)\|^p_{\mathcal{L}(X_1,X_2)}dt\right)^{1/p} < \infty\};\ 1\le p<\infty. \\ P^\infty(\Omega;\mathcal{L}(X_1,X_2)) &:= \{F\in P(\Omega;\mathcal{L}(X_1,X_2)) \mid \|F\|_\infty := \\ &\qquad \operatorname{ess\,sup}_\Omega \|F(t)\|_{\mathcal{L}(X_1,X_2)} < \infty\}. \end{aligned}$$

$$\begin{aligned} L(\Omega;X) &:= \{f:\Omega\to X \mid \langle x, f(\cdot)\rangle \text{ is measurable for all } x\in X\}. \\ L^p(\Omega;X) &:= \{f\in L(\Omega;X) \mid \|f\|_p := \left(\int_\Omega \|f(t)\|^p_X dt\right)^{1/p} < \infty\};\ 1\le p<\infty. \\ L^\infty(\Omega;X) &:= \{f\in L(\Omega;X) \mid \|f\|_\infty := \operatorname{ess\,sup}_\Omega \|f(t)\|_X < \infty\}. \end{aligned}$$

The reason for using the "L" notation is that these integrals are also defined in the Lebesgue sense. For example, if $(T(t))_{t\ge0}$ is a strongly continuous semigroup, then $T(\cdot)x \in L^p([0,\tau];X)$ for all $x\in X$, but we only have that $T(\cdot)\in P^p([0,\tau];\mathcal{L}(X))$ instead of the Lebesgue space $L^p([0,\tau];\mathcal{L}(X))$ (see Example A.1.11).

We remark that if X_1 and X_2 are finite-dimensional, then $\mathcal{L}(X_1,X_2)$ is also finite-dimensional, and so $L^\infty(\Omega;\mathcal{L}(X_1,X_2))$ is well-defined as a Lebesgue space (see Lemma A.1.6) and equals $P^\infty(\Omega;\mathcal{L}(X_1,X_2))$.

Lemma A.1.15. *If we do not distinguish between two functions that differ on a set of measure zero, then the spaces $P^p(\Omega;\mathcal{L}(X_1,X_2))$, $P^\infty(\Omega;\mathcal{L}(X_1,X_2))$, $L^p(\Omega;X)$, and $L^\infty(\Omega;X)$ are Banach spaces.*

Furthermore, $L^2(\Omega;X)$ is a Hilbert space with inner product

$$\langle h,f\rangle = \int_\Omega \langle h(t), f(t)\rangle_X\,dt. \tag{A.4}$$

Proof. See Thomas [53] or [52]. The completeness property of L^p is also shown in theorem III.6.6 of Dunford and Schwartz [13]. In Section 3.5 of Balakrishnan [5] it is shown that $L^2(\Omega,X)$ is a Hilbert space. □

A.2 The Hardy spaces

In this section, we consider some special classes of functions that are holomorphic on the open half-plane $\mathbb{C}_0^+ := \{s\in\mathbb{C} \mid \operatorname{Re}(s)>0\}$.

Good general references for this section are Kawata [31] and Helson [23] for the scalar case and Thomas [53] and Rosenblum and Rovnyak [49] for the vector-valued case.

As in finite dimensions, we define holomorphicity of a complex-valued function as differentiability.

Definition A.2.1. Let X_1 and X_2 be Hilbert spaces, and let $F : \Upsilon \to \mathcal{L}(X_1, X_2)$, where Υ is a domain in $\mathbb{C}$. Then F is *holomorphic* on Υ if F is weakly differentiable on Υ, i.e., for all $x_1 \in X_1, x_2 \in X_2$, the function $\langle F(\cdot)x_1, x_2\rangle$ is differentiable.

Example A.2.2. Let A be a closed linear operator on the Hilbert space X. Define $F : \rho(A) \to \mathcal{L}(X)$ by $F(\lambda) = (\lambda I - A)^{-1}$. We shall prove that F is holomorphic on $\rho(A)$. The resolvent equation (5.12) implies

$$\langle((\lambda + h)I - A)^{-1}x_1, x_2\rangle - \langle(\lambda I - A)^{-1}x_1, x_2\rangle \\ = \langle -h(\lambda I - A)^{-1}((\lambda + h)I - A)^{-1}x_1, x_2\rangle.$$

Since F is continuous, this implies that F is weakly differentiable with $\frac{dF}{d\lambda}(\lambda) = -(\lambda I - A)^{-2}$. Thus the resolvent operator is holomorphic.

Our definition can be seen as "weak holomorphic". We remark that uniform and weak holomorphicity are equivalent. Next we define special classes of holomorphic functions.

Definition A.2.3. For a Banach space $\mathcal{W}$ and a separable Hilbert space X we define the following *Hardy spaces*:

$$\mathbf{H}^\infty(\mathcal{W}) := \left\{ G : \mathbb{C}_0^+ \to \mathcal{W} \mid G \text{ is holomorphic and } \sup_{\mathrm{Re}(s)>0} \|G(s)\| < \infty \right\};$$

$$\mathbf{H}^2(X) := \Bigg\{ f : \mathbb{C}_0^+ \to X \mid f \text{ is holomorphic and} \\ \|f\|_2^2 = \sup_{\zeta>0} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\zeta + i\omega)\|^2 d\omega \right) < \infty \Bigg\}. \tag{A.5}$$

When the Banach space $\mathcal{W}$ or the Hilbert space X equals $\mathbb{C}$, we shall use the notation $\mathbf{H}^\infty$ and $\mathbf{H}^2$ for $\mathbf{H}^\infty(\mathbb{C})$ and $\mathbf{H}^2(\mathbb{C})$, respectively. In most of the literature, Hardy spaces on the disc are usually treated; see, for example, Rosenblum and Rovnyak [49].

Lemma A.2.4. *If $\mathcal{W}$ is a Banach space, then $\mathbf{H}^\infty(\mathcal{W})$ from Definition* A.2.3 *is a Banach space under the $\mathbf{H}^\infty$-norm*

$$\|G\|_\infty := \sup_{Re(s)>0} \|G(s)\|_{\mathcal{W}}. \tag{A.6}$$

Proof. See Theorem D of Rosenblum and Rovnyak [49, section 4.7]. □

We now collect several important results in the following lemma.

Lemma A.2.5. *The following are important properties of* $\mathbf{H}^\infty(\mathcal{L}(U,Y))$*, where* U, Y *are separable Hilbert spaces:*

1. *For every* $F \in \mathbf{H}^\infty(\mathcal{L}(U,Y))$ *there exists a unique function*
$$\tilde{F} \in P^\infty((-i\infty, i\infty); \mathcal{L}(U,Y))$$
such that
$$\lim_{x \downarrow 0} F(x+i\omega)u = \tilde{F}(i\omega)u \qquad \textit{for all } u \in U \textit{ and almost all } \omega \in \mathbb{R}$$
(i.e., $F \in \mathbf{H}^\infty(\mathcal{L}(U,Y))$ *has a well-defined extension to the boundary);*
2. *The mapping* $F \mapsto \tilde{F}$ *is linear, injective and norm preserving, i.e.,*
$$\sup_{\mathrm{Re}(s)>0} \|F(s)\|_{\mathcal{L}(U,Y)} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \|\tilde{F}(i\omega)\|_{\mathcal{L}(U,Y)}$$
(consequently, we can identify $F \in \mathbf{H}^\infty(\mathcal{L}(U,Y))$ *with its boundary function* $\tilde{F} \in P^\infty((-i\infty, i\infty); \mathcal{L}(U,Y))$ *and we can regard* $\mathbf{H}^\infty(\mathcal{L}(U,Y))$ *as a closed subspace of the Banach space* $P^\infty((-i\infty, i\infty); \mathcal{L}(U,Y))$*);*
3. *Identifying* F *with* $\tilde{F}$*, the following holds:*
$$\sup_{\mathrm{Re}(s)\geq 0} \|F(s)\|_{\mathcal{L}(U,Y)} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \|F(i\omega)\|_{\mathcal{L}(U,Y)} < \infty.$$

Proof. See theorems A of sections 4.6 and 4.7 of Rosenblum and Rovnyak [49]. □

We remark that Rosenblum and Rovnyak [49] use the notation L^∞ for P^∞. In general, the boundary function $\tilde{F}$ will not have the property that $\tilde{F}$ is uniformly measurable in the $\mathcal{L}(U,Y)$ topology; see Rosenblum and Rovnyak [49, exercise 1 of chapter 4] or Thomas [53].

Lemma A.2.6. $\mathbf{H}^2(X)$ *is a Banach space under the* $\mathbf{H}^2$*-norm defined by* (A.5)*, and the following important properties hold:*

1. *For each* $f \in \mathbf{H}^2(X)$ *there exists a unique function* $\tilde{f} \in L^2((-i\infty, i\infty); X)$ *such that*
$$\lim_{x \downarrow 0} f(x+i\omega) = \tilde{f}(i\omega) \qquad \textit{for almost all } \omega \in \mathbb{R}$$
and
$$\lim_{x \downarrow 0} \|f(x+\cdot) - \tilde{f}(\cdot)\|_{L^2((-i\infty,i\infty);X)} = 0;$$
2. *The mapping* $f \to \tilde{f}$ *is linear, injective, and* $\|f\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\tilde{f}(i\omega)\|^2 d\omega$*, i.e., it is norm preserving*

   (consequently, we identify the function $f \in \mathbf{H}^2(X)$ *with its boundary function* $\tilde{f} \in L^2((-i\infty, i\infty); X)$ *and regard* $\mathbf{H}^2(X)$ *as a closed subspace of* $L^2((-i\infty, i\infty); X)$*);*

3. *Denote by* $\mathbb{C}_\alpha^+$ *the subset* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$. *For any* $f \in \mathbf{H}^2(X)$ *and any* $\alpha > 0$ *we have that*

$$\lim_{\rho\to\infty}\left(\sup_{s\in\overline{\mathbb{C}_\alpha^+};\ |s|>\rho} \|f(s)\| \right) = 0 \tag{A.7}$$

(sometimes the terminology $f(s) \to 0$ *as* $|s| \to \infty$ *in* $\overline{\mathbb{C}_\alpha^+}$ *is used).*

Proof. Parts 1 *and* 2. The proof for the scalar case as given by Kawata [31, theorem 6.5.1]. Since this theorem holds for vector-valued function as well, the proof of parts a and b is similar to that for the scalar case.

Part 3. See Hille and Phillips [24, theorem 6.4.2]. □

We remark that in general part 3 is not true for $\alpha = 0$. ¿From this lemma we deduce the following result.

Corollary A.2.7. *If* X *is a separable Hilbert space, then* $\mathbf{H}^2(X)$ *is a Hilbert space under the inner product*

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle f(i\omega), g(i\omega)\rangle d\omega.$$

$\mathbf{H}^2(X)$ is a very special Hilbert space, as is apparent from the following lemma and the Paley-Wiener theorem.

Lemma A.2.8. *Let* X *be a separable Hilbert space and let* $f \in \mathbf{H}^2(X)$ *be different from the zero function. Then* f *is nonzero almost everywhere on the imaginary axis.*

Proof. Suppose that there is a subset V of the imaginary axis with positive measure such that f is zero on this set. Then for every $x \in X$, we have that $\langle f, x\rangle \in \mathbf{H}^2$ and it is zero on V. This implies that

$$\int_{-\infty}^{\infty} \frac{|\log(\langle f(i\omega), x\rangle)|}{1+\omega^2} d\omega = \infty.$$

By Theorem 6.6.1 of Kawata [31] this can only happen if $\langle f, x\rangle$ is the zero function. Since $x \in X$ was arbitrary, this would imply that $f = 0$. This is in contradiction to our assumption, and so the set V cannot have positive measure. □

Theorem A.2.9 (Paley-Wiener Theorem)**.** *If* X *is a separable Hilbert space, then under the Laplace transform* $L^2([0,\infty); X)$ *is isomorphic to* $\mathbf{H}^2(X)$ *and it preserves the inner products.*

Proof. See Thomas [53]. □

The following theorem gives a characterization of bounded operators between frequency-domain spaces.

Theorem A.2.10. *Suppose that U and Y are separable Hilbert spaces.*

1. *If $F \in P^\infty((-i\infty, i\infty); \mathcal{L}(U,Y))$ and $u \in L^2((-i\infty, i\infty); U)$, then $Fu \in L^2((-i\infty, i\infty); Y)$. Moreover, the multiplication map $\Lambda_F : u \mapsto Fu$ defines a bounded linear operator from $L^2((-i\infty, i\infty); U)$ to $L^2((-i\infty, i\infty); Y)$, and*

$$\|\Lambda_F u\|_{L^2((-i\infty,i\infty);Y)} \leq \|F\|_\infty \|u\|_{L^2((-i\infty,i\infty);U)},$$

where $\|\cdot\|_\infty$ denotes the norm on $P^\infty((-i\infty, i\infty); \mathcal{L}(U,Y))$. In fact,

$$\|\Lambda_F\| = \sup_{u \neq 0} \frac{\|\Lambda_F u\|_{L^2((-i\infty,i\infty);Y)}}{\|u\|_{L^2((-i\infty,i\infty);U)}} = \|F\|_\infty.$$

2. *If $F \in \mathbf{H}^\infty(\mathcal{L}(U,Y))$ and $u \in \mathbf{H}^2(U)$, then $Fu \in \mathbf{H}^2(Y)$. Moreover, the multiplication map $\Lambda_F : u \mapsto Fu$ defines a bounded linear operator from $\mathbf{H}^2(U)$ to $\mathbf{H}^2(Y)$, and*

$$\|\Lambda_F u\|_{\mathbf{H}^2(Y)} \leq \|F\|_\infty \|u\|_{\mathbf{H}^2(U)},$$

where $\|\cdot\|_\infty$ denotes the norm on $\mathbf{H}_\infty(\mathcal{L}(U,Y))$. In fact,

$$\|\Lambda_F\| = \sup_{u \neq 0} \frac{\|\Lambda_F u\|_{\mathbf{H}^2(Y)}}{\|u\|_{\mathbf{H}^2(U)}} = \|F\|_\infty.$$

3. *$F \in P^\infty((-i\infty, i\infty); \mathcal{L}(U,Y))$ is in $\mathbf{H}^\infty(\mathcal{L}(U,Y))$ if and only if $\Lambda_F \mathbf{H}^2(U) \subset \mathbf{H}^2(Y)$.*

Proof. *Part* 1. See Thomas [53].

Part 2. It is easy to show that for $F \in \mathbf{H}^\infty(\mathcal{L}(U,Y))$ the first inequality holds. So $\|\Lambda_F\| \leq \|F\|_\infty$. To prove the other inequality, let $\lambda \in \mathbb{C}_0^+$, $y_0 \in Y$ and $f \in \mathbf{H}^2(U)$. Consider

$$\begin{aligned}
\left\langle f, \Lambda_F^* \frac{y_0}{\cdot + \lambda} \right\rangle_{\mathbf{H}^2(U)} &= \left\langle \Lambda_F f, \frac{y_0}{\cdot + \lambda} \right\rangle_{\mathbf{H}^2(Y)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\langle F(i\omega) f(i\omega), \frac{y_0}{i\omega + \lambda} \right\rangle_Y d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle F(i\omega) f(i\omega), y_0 \rangle_Y \frac{-1}{i\omega - \overline{\lambda}} d\omega \\
&= \langle F(\overline{\lambda}) f(\overline{\lambda}), y_0 \rangle_Y \qquad \text{by Cauchy's Theorem} \\
&= \langle f(\overline{\lambda}), F(\overline{\lambda})^* y_0 \rangle_U \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle f(i\omega), F(\overline{\lambda})^* y_0 \rangle_U \frac{-1}{i\omega - \overline{\lambda}} d\omega \\
&\qquad \text{using Cauchy's Theorem again}
\end{aligned}$$

$$= \langle f, F(\overline{\lambda})^* \frac{y_0}{\cdot + \lambda} \rangle_{\mathbf{H}^2(U)}.$$

Since the above equality holds for every $f \in \mathbf{H}^2(U)$, we have that

$$\Lambda_F^* \frac{y_0}{\cdot + \lambda} = F(\overline{\lambda})^* \frac{y_0}{\cdot + \lambda}.$$

This implies that $\|\Lambda_F^*\| \geq \|F^*\|_\infty$. Now the general property that $\|F^*\|_\infty = \|F\|_\infty$, concludes the proof.

Part 3. See Thomas [53]. □

The proof of part 2 was communicated by George Weiss.

Bibliography

[1] B.D.O. Anderson. A system theory criterion for positive real matrices. *SIAM J. Control Optim.*, 5:171–182, 1967.

[2] B.D.O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Prentice-Hall electrical engineering series, Networks series. Englewood Cliffs, NJ: Prentice-Hall, 1973. Republished by Dover Publcations in 1996, ISBN 048645357X.

[3] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[4] A. Baaiu, F. Couenne, D. Eberard, C. Jallut, L. Lefèvre, Y. Le Gorrec, and B. Maschke. Port-based modelling of mass transport phenomena. *Math. Comput. Model. Dyn. Syst.*, 15(3):233–254, 2009.

[5] A.V. Balakrishnan. *Applied Functional Analysis*. Springer-Verlag, New York, 1976.

[6] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and Control of Infinite Dimensional Systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, second edition, 2007.

[7] J. Cervera, A. J. van der Schaft, and A. Baños. Interconnection of port-Hamiltonian systems and composition of Dirac structures. *Automatica J. IFAC*, 43(2):212–225, 2007.

[8] S. Cox and E. Zuazua. The rate at which energy decays in a string damped at one end. *Indiana Univ. Math. J.*, 44(2):545–573, 1995.

[9] R. F. Curtain and A. J. Pritchard. *Infinite Dimensional Linear Systems Theory*, volume 8 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 1978.

[10] R.F. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

B. Jacob and H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*,
Operator Theory: Advances and Applications 223, DOI 10.1007/978-3-0348-0399-1,

[11] W. Desch and W. Schappacher. Spectral properties of finite-dimensional perturbed linear semigroups. *J. Differential Equations*, 59(1):80–102, 1985.

[12] J. Diestel and J.J. Uhl, Jr. *Vector Measures.* American Mathematical Society, Providence, R.I., 1977.

[13] N. Dunford and J.T. Schwartz. *Linear Operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Reprint of the 1958 original, A Wiley-Interscience Publication.

[14] D. Eberard, B. M. Maschke, and A. J. van der Schaft. An extension of Hamiltonian systems to the thermodynamic phase space: towards a geometry of nonreversible processes. *Rep. Math. Phys.*, 60(2):175–198, 2007.

[15] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000.

[16] H. Fattorini. Boundary control systems. *SIAM Journal on Control*, Jan 1968.

[17] H. O. Fattorini. *Infinite-dimensional optimization and control theory*, volume 62 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1999.

[18] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of linear operators. Vol. I*, volume 49 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1990.

[19] V.I. Gorbachuk and M.L. Gorbachuk. *Boundary Value Problems for Operator Differential Equations*, volume 48 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1991. Translated and revised from the 1984 Russian original.

[20] K. Gustafson and G. Lumer. Multiplicative perturbation of semigroup generators. *Pacific J. Math.*, 41:731–742, 1972.

[21] B. Hamroun, A. Dimofte, L. Lefèvre, and E. Mendes. Control by interconnection and energy-shaping methods of port Hamiltonian models. Application to the shallow water equations. *Eur. J. Control*, 16(5):545–563, 2010.

[22] M.L.J. Hautus. Controllability and observability conditions of linear autonomous systems. *P. K. Ned Akad A Math*, 72(5):443–448, Jan 1969.

[23] H. Helson. *Harmonic Analysis.* Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1983.

[24] E. Hille and R.S. Phillips. *Functional Analysis and Semi-groups.* American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

[25] D. Hinrichsen and A.J. Pritchard. *Mathematical Systems Theory. I*, volume 48 of *Texts in Applied Mathematics*. Springer-Verlag, Berlin, 2005.

[26] F.L. Huang. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. Differential Equations*, 1(1):43–56, 1985.

[27] C.A. Jacobson and C.N. Nett. Linear state-space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans. Automat. Control*, 33(6):541–549, 1988.

[28] D. Jeltsema and A. J. van der Schaft. Lagrangian and Hamiltonian formulation of transmission line systems with boundary energy flow. *Rep. Math. Phys.*, 63(1):55–74, 2009.

[29] R.E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana (2)*, 5:102–119, 1960.

[30] T. Kato. *Perturbation Theory for Linear Operators*. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer-Verlag New York, Inc., New York, 1966.

[31] T. Kawata. *Fourier Analysis in Probability Theory*. Academic Press, New York, 1972.

[32] H.-W. Knobloch and H. Kwakernaak. *Lineare Kontrolltheorie*. Akademie-Verlag, Berlin, 1986.

[33] M. Kurula, H. Zwart, A. van der Schaft, and J. Behrndt. Dirac structures and their composition on Hilbert spaces. *J. Math. Anal. Appl.*, 372(2):402–422, 2010.

[34] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I*, volume 74 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000.

[35] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. II*, volume 75 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000.

[36] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. Control Optim.*, 44(5):1864–1892 (electronic), 2005.

[37] J.-L. Lions. *Optimal control of systems governed by partial differential equations*. Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York, 1971.

[38] J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications. Vol. II.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[39] G. Lumer and R.S. Phillips. Dissipative operators in a Banach space. *Pacific J. Math.*, 11:679–698, 1961.

[40] Z-H. Luo, B-Z. Guo, and O. Morgul. *Stability and Stabilization of Infinite Dimensional Systems with Applications.* Communications and Control Engineering Series. Springer-Verlag London Ltd., London, 1999.

[41] A. Macchelli and C. Melchiorri. Control by interconnection of mixed port Hamiltonian systems. *IEEE Trans. Automat. Control*, 50(11):1839–1844, 2005.

[42] S.A. Nefedov and F.A. Sholokhovich. A criterion for stabilizability of dynamic systems with finite-dimensional input. *Differentsial'nye Uravneniya*, 22(2):223–228, 364, 1986.

[43] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica J. IFAC*, 38(4):585–596, 2002.

[44] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[45] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[46] R.S. Phillips. Dissipative operators and hyperbolic systems of partial differential equations. *Trans. Amer. Math. Soc.*, 90:193–254, 1959.

[47] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory, A Behavioral Approach*, volume 26 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1998.

[48] R. Rebarber and H. Zwart. Open-loop stabilizability of infinite-dimensional systems. *Math. Control Signals Systems*, 11(2):129–160, 1998.

[49] M. Rosenblum and J. Rovnyak. *Hardy Classes and Operator Theory.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1985.

[50] E.D. Sontag. *Mathematical Control Theory*, volume 6 of *Texts in Applied Mathematics.* Springer-Verlag, New York, second edition, 1998.

[51] O. Staffans. *Well-posed Linear Systems*, volume 103 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2005.

[52] E.G.F. Thomas. Totally summable functions with values in locally convex spaces. In *Measure theory (Proc. Conf., Oberwolfach, 1975)*, pages 117–131. Lecture Notes in Math., Vol. 541. Springer, Berlin, 1976.

[53] E.G.F. Thomas. Vector-valued integration with applications to the operator-valued H^∞ space. *IMA J. Math. Control Inform.*, 14(2):109–136, 1997.

[54] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[55] A. van der Schaft. *L_2-gain and Passivity Techniques in Nonlinear Control.* Communications and Control Engineering Series. Springer-Verlag London Ltd., London, second edition, 2000.

[56] A. van der Schaft. Port-Hamiltonian systems: an introductory survey. In *International Congress of Mathematicians. Vol. III*, pages 1339–1365. Eur. Math. Soc., Zürich, 2006.

[57] A. van der Schaft and B. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *J. Geom. Phys.*, 42(1–2):166–194, 2002.

[58] J.A. Villegas. *A Port-Hamiltonian Approach to Distributed Parameter Sy stems.* PhD thesis, Department of Applied Mathematics, Enschede, The Netherlands, May 2007. Availible at http://doc.utwente.nl.

[59] J.A. Villegas, H. Zwart, Y. Le Gorrec, and B. Maschke. Exponential stability of a class of boundary control systems. *IEEE Trans. Automat. Control*, 54(1):142–147, 2009.

[60] G. Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems (MCSS)*, 7(1):23–57, 1994.

[61] K. Yosida. *Functional Analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, sixth edition, 1980.

[62] J. Zabczyk. A note on C_0-semigroups. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 23(8):895–898, 1975.

[63] H. Zwart. *Geometric Theory for Infinite-Dimensional Systems*, volume 115 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, Berlin, 1989.

[64] H. Zwart. Transfer functions for infinite-dimensional systems. *Systems Control Lett.*, 52(3-4):247–255, 2004.

[65] H. Zwart, Y. Le Gorrec, B. Maschke, and J. Villegas. Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 16(4):1077–1093, Oct 2010.

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